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Asymptotic Estimates for Some Dispersive Equations on the Alpha-modulation Space

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ABSTRACT

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The α-modulation space, $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$, is a function space developed by Gröbner in 1992. The α-modulation space is a generalization of the modulation space, $M_{p,q}^{s}(\mathbb{R}^n)$, and Besov space, $B_{p,q}^{s}(\mathbb{R}^n)$. In this thesis we obtain asymptotic estimates for the Cauchy Problem for dispersive equation, a generalized half Klein-Gordon, and the Klein-Gordon equations. The wave equations will also be considered in this thesis too. These estimates were found by using standard tools from harmonic analysis. Then we use these estimates with a multiplication algebra property of the α-modulation space to prove that there are unique solutions locally in time for a nonlinear version of these partial differential equations in the function space $C([0,T], M_{p,q}^{s,\alpha}(\mathbb{R}^n))$. These results are obtained by using the fixed point theorem.

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Chapter 1

Background

1.1 Introduction

Nonlinear dispersive equations have been studied both in mathematics and physics due to their importance in describing how waves of different wavelength propagate at different velocities. The Schrödinger equation, Airy and Klein-Gordon equation are classic examples of dispersive equations. We will also consider the wave equation in this thesis too. An intuitive description of what dispersive equations describe would be the following: as the main wave travels through a medium it will begin to break into many smaller waves as time evolves. As time continues to evolve, the waves of different wavelengths propagate at different velocities. A quantitative understanding of this behavior gives one insight into the behavior of the solution.

Since very few partial differential equations (PDEs) can be solved explicitly other methods need to be used to establish the existence of a solution. Thus additional techniques are required to understand when a PDE has a solution and how such a solution behaves. Some of these analytical techniques are: separation of variables, method of characteristics, change of variables, the Lie group method, and numerical methods. In this thesis we are going to focus on one analytical method that would be classified as an integral transform. Our integral transform relies on using the Fourier transform to turn the PDE into a differential equation that is easier to solve explicitly. In this process Fourier multipliers are produced.

As mentioned above, Fourier multipliers arise naturally in the study of PDEs and in the convergence of Fourier series. The fundamental problem in the study of Fourier multipliers is to relate the boundness properties of the Fourier multiplier on certain function spaces to properties of its symbol. In 1977 Strichartz [25] found $L^p(\mathbb{R}^n)$ space decay estimates for the solutions to the Schrödinger equation and the Klein-Gordon equation with certain conditions on the initial data. In 1995 both Lindblad and Sogge [21], and Ginibre and Velo [13] found non-endpoint estimates for the wave equation. Shortly after that in 1998 Keel and Tao [19] found endpoint estimates for the Schrödinger and wave equation.

In general unimodular Fourier multipliers do not preserve any L^p-norm except for when $p = 2$. For this reason $L^p(\mathbb{R}^n)$ spaces may not be an appropriate space for this study. In 1983 Feichtinger [8] introduced a new function space called the modulation space, denoted $M^s_{p,q}(\mathbb{R}^n)$. The modulation space is used to measure the smoothness of a function or distribution in a different way than the $L^p(\mathbb{R}^n)$ space, and can be understood as a measure of the phase space concentration of a function. This has made it possible for other Strichartz estimates to be found.

In 2007 Benyi and Grochenig [2] showed that the solutions to both the Schrödinger equation and the Wave equation are in the non-weighted modulation space $M^0_{p,q}(\mathbb{R}^n)$ provided their initial data is in $M^0_{p,q}(\mathbb{R}^n)$. Wang and Hudzik [27] studied the nonlinear Schrödinger equation and the nonlinear Klein-Gordon equation. Supposing that the initial data is in the non-weighted modulation space, $M^0_{p,q}(\mathbb{R}^n)$, using specified nonhomogenuous functions they were able to show that both the nonlinear Schrödinger equation and the nonlinear Klein-Gordon equation have solutions in a similar non-weighted modulation space. More general results were found by Chen, Fan, and Sun [5] in 2012 and by Deng, Ding, and Sun [6] in 2013.

Gröbner's Ph.D thesis [15], written in 1992, introduced another function space called the α-modulation space, denoted by $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. This function space was not only a generalization for the modulation space $M^s_{p,q}(\mathbb{R}^n)$, but also served as an intermediate space for both the modulation space $M^s_{p,q}(\mathbb{R}^n)$ and the Besov space, $B^s_{p,q}(\mathbb{R}^n)$. The parameter $\alpha \in [0, 1]$ serves as a tuner in the sense that as α approaches 0^+ the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ begins to "look" more like modulation space

$M^s_{p,q}(\mathbb{R}^n)$. As α approaches 1^- the α-modulation space, $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ begins to "look" more like the Besov space $B^s_{p,q}(\mathbb{R}^n)$. Because of this fact, and the fact that Fourier multipliers can preserve the $M^{s,\alpha}_{p,q}$-norm in more general cases this makes the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ a more suitable function space to work on.

In the following sections we will present basic equations and definitions that will be used throughout this dissertation. The goal is to provide a simple introduction and reference that will serve as a basis for the remainder of the chapters. Other important notations used throughout this dissertation are as follows: define $\mathbb{Z}_* = \mathbb{Z}_+ \cup \{0\}$, for a space $\mathbb{X}$ let $\mathbb{X}'$ denote the dual space of $\mathbb{X}$, for $p \in \mathbb{R}_+$ denote the conjugate of p as p', that is

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the modifications that if $p = 0$, then $p' = \infty$ or if $p = \infty$, then $p' = 0$. For all multi-indices α' define $|\alpha'| = \alpha_1 + \cdots + \alpha_n$, for all $x, \xi \in \mathbb{R}^n$ define $x\xi = x_1\xi_1 + \cdots + x_n\xi_n$, we write $A \preceq B$ if there is a positive constant C such that $A \leq CB$, and define $\langle x \rangle = (1 + |x|)^{\frac{1}{2}}$.

1.2 Fourier Transform

Let $L^p(\mathbb{R}^n)$ denote the usually L^p function space with the norm

$$||f||_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

where $0 < p < \infty$. When $p = \infty$ we will make the usual modification of

$$||f||_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Suppose $\alpha' = (\alpha'_1, \cdots, \alpha'_n)$ and $\beta' = (\beta'_1, \cdots, \beta'_n)$, where $\alpha'_i, \beta'_i \in \mathbb{Z}_*$ for all $1 \leq i \leq n$, are multi-indices. Also, suppose the conventions of $x^{\alpha'} = x_1^{\alpha'_1} \cdots x_n^{\alpha'_n}$ and $\partial^{\beta'} f(x) = \partial^{\beta'_1}_{x_1} \cdots \partial^{\beta'_n}_{x_n} f(x)$. The *Schwartz Space*, denoted by $\mathscr{S}(\mathbb{R}^n)$, is the set of all smooth functions $f(x)$, meaning functions that are infinitely differentiable, that satisfy the following

$$\sup_{x \in \mathbb{R}^n} \left| x^{\alpha'} \partial^{\beta'} f(x) \right| < C_{\alpha', \beta'} < \infty,$$

where $C_{\alpha',\beta'}$ is a constant that depends on all multi-indices α' and β'. The Schartz space can be thought of as the set of all smooth functions that decay faster than the reciprocal of any polynomial. With this view point it is clear that $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

For all $f \in \mathscr{S}(\mathbb{R}^n)$ define the *Fourier transform*, denoted by $\hat{f}(\xi) = \mathscr{F}f(\xi)$, by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx, \tag{1.1}$$

where $\xi \in \mathbb{R}^n$ and $x\xi = x_1\xi_1 + \cdots + x_n\xi_n$. Since $\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ equation (1.1) makes sense for all $f \in \mathscr{S}(\mathbb{R}^n)$. Furthermore, the Fourier transform enjoys the linear property

$$\widehat{af+bg}(\xi) = a\hat{f}(\xi) + b\hat{g}(\xi),$$

where $f, g \in \mathscr{S}(\mathbb{R}^n)$ and $a, b \in \mathbb{C}$, and enjoys the following other properties as well:

1. $\hat{\tilde{f}} = \tilde{\hat{f}}$,
2. $\widehat{\tau^y(f)}(\xi) = e^{-iy\xi}\hat{f}(\xi)$,
3. $(e^{ixy}f(x))\hat{}\,(\xi) = \tau^y(\hat{f})(\xi)$,
4. $(\delta^a f)\hat{}\,(\xi) = t^{-n}\delta^{t^{-1}}(\hat{f})$,
5. $(\partial^{\alpha'} f)\hat{}\,(\xi) = (i\xi)^{\alpha'}\hat{f}(\xi)$,
6. $(\partial^{\alpha'}\hat{f})(\xi) = ((-ix)^{\alpha'}f(x))\hat{}\,(\xi)$,
7. $\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$,

with $a > 0$, $t > 0$, and $\tau^y(f)(x), \delta^a f(x), \tilde{f}(x)$ and $f * g(x)$ are defined as

$$\tau^y(f)(x) = f(x-y), \tag{1.2}$$

$$\delta^a f(x) = f(ax), \tag{1.3}$$

$$\tilde{f}(x) = f(-x), \tag{1.4}$$

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{1.5}$$

See [14, 20] for details of the proofs of these properties.

To define the *inverse of the Fourier transform*, first note that if $f \in \mathscr{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathscr{S}(\mathbb{R}^n)$ [14]. Because of this fact, we are able to define the inverse of the Fourier transform for all $f \in \mathscr{S}(\mathbb{R}^n)$. Define the inverse $\check{f}(x) = \mathscr{F}^{-1} f(x)$ by

$$\mathscr{F}^{-1} f(x) = \check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi) e^{ix\xi} d\xi. \tag{1.6}$$

With this definition we have the following Fourier inversion property:

$$\mathscr{F}^{-1}\left(\mathscr{F}(f)(\xi)\right)(x) = \mathscr{F}\left(\mathscr{F}^{-1}(f)(\xi)\right)(x) = f(x).$$

We also have the important Plancherel's Identity.

Theorem 1.2.1. *(Plancherel's Identity [14]) For all $f \in \mathscr{S}(\mathbb{R}^n)$ the following holds*

$$||f||_{L^2(\mathbb{R}^n)} = \left|\left|\hat{f}\right|\right|_{L^2(\mathbb{R}^n)} = ||\check{f}||_{L^2(\mathbb{R}^n)}.$$

Note that Plancherel's Identity is often associated with a conservation of energy estimate.

We are able to define the Fourier transform on $L^1(\mathbb{R}^n)$. This follows immediately because if $f \in L^1(\mathbb{R})$ then

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx,$$

converges and makes sense. Decay and continuity of such Fourier transforms of $L^1(\mathbb{R}^n)$ functions are characterized in the following theorem:

Theorem 1.2.2. *(Riemann-Lebesgue Lemma [16, 14, 20]) If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is uniformly continuous and*

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

This result can also be interpreted as a smoothing effect that the Fourier transform has on L^1-functions, which may not be continuous.

Note that we are not always able to define the inversion. To see this, consider the following example:

Example 1.2.3. ([20]) Let $n = 1$ and set $f(x) = \chi_{(a,b)}(x)$ to be the characteristic function on the interval (a, b). Clearly $f \in L^1(\mathbb{R})$. Then it follows that

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx \\ &= \int_{a}^{b} e^{-ix\xi}dx \\ &= -\frac{e^{-ib\xi} - e^{-ia\xi}}{i\xi} \\ &= -e^{-i(a+b)\xi}\frac{\sin((b-a)\xi)}{\xi}.\end{aligned}$$

Here we see $\hat{f} \notin L^1(\mathbb{R})$.

Because of this, the Fourier inverse is defined only when it makes sense, that is, the integral is convergent in the L^1-sense; or when $\hat{f} \in L^1(\mathbb{R}^n)$.

We can also extend the definition of the Fourier transform to $L^2(\mathbb{R}^n)$ by considering a dense subspace of $L^2(\mathbb{R}^n)$. Such a space would be $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Using Theorem 1.2.1, to define the Fourier transform of $f \in L^2(\mathbb{R}^n)$ pick a sequence $f_N \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that converges to f in $L^2(\mathbb{R}^n)$. Since $f_N \in L^1 \cap L^2$, then $\hat{f}_N$ will be defined for all N. Then it follows that the function $\hat{f}$ such that

$$\left|\left|\hat{f}_N - \hat{f}\right|\right|_{L^2(\mathbb{R}^n)} \to 0 \text{ as } N \to \infty,$$

can be taken as the Fourier transform of f. The existence of $\hat{f} \in L^2(\mathbb{R}^n)$ is guaranteed since the Fourier Transform is an isometry on $L^2(\mathbb{R}^n)$ [14]. Note that the definition of $\hat{f}$ is independent from the choice of f_N. See [14, 20].

The final extension we will define is the Fourier transform for all $L^p(\mathbb{R}^n)$ where $1 < p < 2$. This is simply done by decomposing an L^p-function, f, into the sum of an L^1-function, f_1, and an L^2-function, f_2. One such decomposition is set $f_1(x) = f(x)\chi_{\{x\in\mathbb{R}^n:|f(x)|>1\}}(x)$ and $f_2(x) = f(x)\chi_{\{x\in\mathbb{R}^n:|f(x)|\leq 1\}}(x)$. Thus $\hat{f}$ is defined as

$$\hat{f}(\xi) = \hat{f}_1(\xi) + \hat{f}_2(\xi).$$

Note that the definition of $\hat{f}$ is independent of the choice of f_1 and f_2 [14]. Such an extension leads to the following well-know theorem:

Theorem 1.2.4. *(Hausdorff-Young inequality [14, 20]) For $1 \leq p \leq 2$ and every function $f \in L^p(\mathbb{R}^n)$ we have the following estimate:*

$$\left|\left|\hat{f}\right|\right|_{L^{p'}(\mathbb{R}^n)} \leq ||f||_{L^p(\mathbb{R}^n)}.$$

From the Hausdorff-Young inequality we get the following well known corollary:

Corollary 1.2.5. *([14, 20]) The following estimate holds*

$$\left|\left|\hat{f}\right|\right|_{L^{\infty}(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)}.$$

This result gives one an upper bound for the supremum of $\hat{f}$ over $\xi \in \mathbb{R}^n$, which is the L^1-norm of f.

1.3 Fourier Multipliers

Let $\mathscr{C}_C^{\infty}(\mathbb{R}^n)$ be the set of all functions that are smooth and have compact support. We call $\mathscr{C}_C^{\infty}(\mathbb{R}^n)$ the set of *Test Functions*. For all $f \in \mathscr{C}_C^{\infty}(\mathbb{R}^n)$ we define the *Fourier multiplier with symbol* μ, denoted $H_\mu f(x)$, by

$$H_\mu f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mu(\xi)\hat{f}(\xi)e^{ix\xi}d\xi. \tag{1.7}$$

First note that $\mathscr{C}_C^{\infty}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$. From this, if $f \in \mathscr{C}_C^{\infty}(\mathbb{R}^n)$, then $H_\mu f$ makes sense. Second, the Fourier multiplier is a linear operator. As we will see later, the Fourier multiplier arises naturally in the formal solution of PDEs with constant coefficients. As mentioned earlier, a fundamental problem in the study of Fourier multipliers is to relate the boundedness properties of H_μ on certain function spaces to properties of the symbol μ. We will now cite a couple well-known theorems about Fourier multipliers.

The next two theorems give sufficient conditions for the $\mu \in L^{\infty}(\mathbb{R}^n)$ and when the operator H_μ is bounded. These results are known as the Hörmander-Mihlin Multiplier Theorem and Marcinkiewicz Multiplier Thoerem.

Theorem 1.3.1. *(The Hörmander-Mihlin Multiplier Theorem on $\mathbb{R}^n$ [14, 3]) Let $\mu(\xi)$ be a complex-valued bounded function on $\mathbb{R}^n/\{0\}$ that satisfies either:*

1. *Mihlin's condition*

$$|\partial^{\alpha}\mu(\xi)| \leq A|\xi|^{-|\alpha|},$$

for all multi-indices $|\alpha| \leq \left[\frac{n}{2}\right] + 1$, *or,*

2. *Hörmander's condition*

$$\sup_{R>0} R^{-n+2|\alpha|} \int_{R<|\xi|<2R} |\partial^{\alpha}\mu(\xi)|^2 \, d\xi \leq A^2 < \infty,$$

for all multi-indices $|\alpha| \leq \left[\frac{n}{2}\right] + 1$,

then for all $1 < p < \infty$ *we have the following estimate*

$$||H_{\mu}||_{L^p \to L^p} \leq C_n \max\left(p, (p-1)^{-1}\right)\left(A + ||\mu||_{L^{\infty}}\right).$$

For the next theorem let $I_j = \left(-2^{j+1}, -2^{j}\right] \cup \left[-2^{j+1}, -2^{j}\right)$ and $R_j = I_{j_1} \times \cdots \times I_{j_n}$ for all $j = (j_1, \cdots, j_n) \in \mathbb{Z}^n$.

Theorem 1.3.2. *(The Marcinkiewicz Multiplier Thoerem on* $\mathbb{R}^n$ *[14]) Let* μ *be a bounded function on* $\mathbb{R}^n$ *that is* $\mathscr{C}^n$, *i.e. functions whose* n^{th} *derivative exists, in all regions* R_j. *Suppose there is a constant* A *such that for all* $k \in \{1, \cdots, n\}$, *all* $j_1, \cdots, j_k \in \{1, \cdots, n\}$, *all* $l_{j_1}, \cdots, l_{j_k} \in \mathbb{Z}$, *and all* $\xi \in I_{l_s}$ *for* $s \in \{1, \cdots, n\} / \{j_1, \cdots, j_k\}$ *we have*

$$\int_{I_{l_{j_1}}} \cdots \int_{I_{l_{j_k}}} \left|\partial^{j}\mu(\xi)\right| d\xi_{j_k} \cdots d\xi_{j_1} \leq A < \infty,$$

then for $1 < p < \infty$ *there is a constant* $C_n < \infty$ *such that*

$$||H_{\mu}||_{L^p \to L^p} \leq C_n \left(A + ||\mu||_{L^{\infty}}\right) \max\left(p, (p-1)^{-1}\right)^{6n}.$$

Both theorems are consequences of Littlewood-Paley theory and dyadic decomposition, see [14, 24] for more details.

In this thesis we will be focused on a special type of Fourier multiplier. The *Unimodular Fourier multiplier* is a Fourier multiplier whose symbol has the form $e^{i\nu(\xi)}$ where ν is a real-valued function. It is important to note that unimodular Fourier multipliers in general do not preserve the L^p-norm, except when $p = 2$. In the case when $p = 2$, one can simply apply Plancherel's identity, Theorem 1.2.1. To better understand this we first need a well known lemma.

Lemma 1.3.3. *([20]) If $t \neq 0$ and $p' \in [1,2]$, then we have $e^{it\Delta} : L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, where $e^{it\Delta}$ is the Fourier Multiplier with the symbol $e^{it|\xi|}$, is continuous and the following estimate holds*

$$\left|\left|e^{it\Delta} f\right|\right|_{L^p(\mathbb{R}^n)} \leq c|t|^{-\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)} \left|\left|f\right|\right|_{L^{p'}(\mathbb{R}^n)}.$$

Now we can consider the following example:

Example 1.3.4. ([20]) Consider the Fourier multiplier $e^{-it\Delta}$ with the symbol $e^{-it|\xi}$, then the Fourier multiplier H_μ is not a bounded operator from $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $p \neq 2$ and $t \neq 0$. If it was a bounded operator from $L^p \to L^p$ it would be bounded for p'. So without loss of generality, suppose $p > 2$. Let $t \neq 0$ and pick $f \in L^{p'}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Then it follows

$$\begin{aligned} \left|\left|f\right|\right|_{L^p(\mathbb{R}^n)} &= \left|\left|e^{it\Delta} e^{-it\Delta} f\right|\right|_{L^p(\mathbb{R}^n)} \\ &\leq c_0 \left|\left|e^{-it\Delta}\right|\right|_{L^p(\mathbb{R}^n)} \\ &\leq c_0 c(t) \left|\left|f\right|\right|_{L^{p'}(\mathbb{R}^n)}, \end{aligned}$$

which is a contradition.

Furthermore, using the same unimodular Fourier multiplier from Example 1.3.4 we have the next example.

Example 1.3.5. Suppose $p > 2$ and $t \neq 0$. Let $g \in L^2(\mathbb{R}^n)$ such that $g \notin L^p(\mathbb{R}^n)$. Define f to be

$$f(x) = H_\mu g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-it|\xi|} \hat{g}(\xi) e^{ix\xi} d\xi.$$

Note that $f \in L^p(\mathbb{R}^n)$. Then the unimodular Fourier multiplier H_{μ_0}, with symbol

$\mu_0(\xi) = e^{it|\xi|}$ is not a bounded operator from $L^2(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ since

$$\begin{aligned}
H_{\mu_0} f(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|} \hat{f}(\xi) e^{ix\xi} d\xi \\
&\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|} \mathscr{F} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-it|\xi|} \hat{g}(\xi) e^{ix\xi} d\xi \right) (x) e^{ix\xi} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|} \mathscr{F} \left(\mathscr{F}^{-1} \left(e^{-it|\xi|} \hat{g}(\xi) \right) \right) (x) e^{ix\xi} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|} e^{-it|\xi|} \hat{g}(\xi) e^{ix\xi} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{ix\xi} d\xi \\
&= g(x).
\end{aligned}$$

and this is not in $L^p(\mathbb{R}^n)$.

Because of these two examples, one sees that L^p spaces may not be appropriate function spaces to study Fourier multipliers and formal solutions to PDEs. Next we will make the precise definition of what we mean by dispersive equations, and look at some dispersive equations that we will be considering in this dissertation.

1.4 Dispersive Equations

Consider the general PDE of the form

$$F(\partial_x, \partial_t) u(t,x) = 0,$$

where F is a polynomial in the partial derivatives. We seek to find a solution to this PDE that takes the form of

$$u(t,x) = Ae^{i(kx - t\xi)}.$$

Note $u(t,x)$ will be a solution if and only if $F(ik, -i\xi) = 0$, which is called a *dispersive relation*. In some cases we can write $\xi = \xi(k)$. With this we can define the *group velocity*, denoted by $c_g(k)$, as

$$c_g(k) = \xi'(k).$$

If c_g is not constant, i.e. $c_g'(k) = \xi''(k) \neq 0$, then the waves are called *dispersive*.

The physical meaning can be interpreted as time evolves waves of different wavelengths disperse in the medium at different velocities. That is a wave with one hump will break into several smaller waves over time. Such equations have been studied in both mathematics and physics because of this property. In this dissertation we will consider both Cauchy problem for dispersive equation, a generalized Klein-Gordon, wave, and the Klein-Gordon equation. First we must introduce the fractional Laplacian.

Let $\Delta = \Delta_x = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ the standard Laplacian. Let $\alpha_o \in \mathbb{R}_+$. Now define the *fractional Laplacian with order* α_o *of* f, denoted by $(-\Delta)^{\frac{\alpha_o}{2}} f(x)$, by

$$(-\Delta)^{\frac{\alpha_o}{2}} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{\alpha_o} \hat{f}(\xi) e^{ix\xi} d\xi. \tag{1.8}$$

Such an operator is formally known as the *Riesz potential*, and has been studied independently. The reader is directed to [23] for more details. With this definition we can state the *Cauchy problem for dispersive equation* as

$$\begin{cases} i\partial_t u(t,x) + |\Delta|^{\frac{\alpha_o}{2}} u(t,x) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{1.9}$$

We can get a formal solution to this PDE by taking the Fourier transform to get an equivalent system of

$$\begin{cases} \partial_t \hat{u}(t,\xi) = i|\xi|^{\alpha_o} \hat{u}(t,\xi), & \text{for } (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), & \text{for } \xi \in \mathbb{R}^n, \end{cases}$$

which is a first order differential equation. Such equation has a solution of the form

$$\hat{u} = e^{it|\xi|^{\alpha_o}} \hat{u}_0(\xi).$$

Now taking the Fourier inverse we have the formal solution to Cauchy problem for dispersive equation (1.9)

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|^{\alpha_o}} \hat{u}_0(\xi) e^{ix\xi} d\xi \\ &= H_{e^{it|\xi|^{\alpha_o}}} u_0(x). \end{aligned}$$

For the sake of convenience we will write the formal solution, $H_{e^{it|\xi|^{\alpha_o}}} u_0(x)$, in the form

$$H_{e^{it|\xi|^{\alpha_o}}} u_0(x) = e^{it|\Delta|^{\frac{\alpha_o}{2}}} u_0(x). \tag{1.10}$$

When $\alpha_o = 1, 2, 3$, the Cauchy problem for dispersive equation corresponds to the (half-) wave equation, the Schrödinger equation, and (essentially) the Airy equation. These cases will be of particular interest, and their solution's asymptotic estimates will be explored in more detail later in the dissertation.

To understand how the wave equation fits into the big picture, first we have

$$\begin{cases} \partial_{tt} u(t,x) - \Delta u(t,x) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_1(x), & \text{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = u_1(x), & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{1.11}$$

and take the Fourier transform we get

$$\begin{cases} \partial_{tt} \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0, \ \text{for } (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n, & \\ \hat{u}(0,\xi) = \hat{u}_1(\xi), & \text{for } \xi \in \mathbb{R}^n, \\ \partial_t \hat{u}(0,\xi) = \hat{u}_1(\xi), & \text{for } \xi \in \mathbb{R}^n. \end{cases}$$

Solving this second order ordinary differential equation we get a solution in the form

$$\hat{u}(t,\xi) = c_1 \cos(t|\xi|) + c_2 \sin(t|\xi|).$$

Using the initial conditions we get

$$\hat{u}_0(\xi) = c_1 \text{ and } \hat{u}_1(\xi) = c_2 |\xi|.$$

Thus we get a solution of

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \cos(t|\xi|) \hat{u}_0(\xi) e^{ix\xi} d\xi + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin(t|\xi|)}{|\xi|} \hat{u}_1(\xi) e^{ix\xi} d\xi \\ &= H_{\cos(t|\xi|)} u_0(x) + H_{\frac{\sin(t|\xi|)}{|\xi|}} u_1(x), \end{aligned}$$

where we can take

$$H_{\cos(t|\xi|)} u_0(x) = \cos(t(-\Delta)^{\frac{1}{2}}) u_0(x) \text{ and } H_{\frac{\sin(t|\xi|)}{|\xi|}} u_1(x) = \frac{\sin(t(-\Delta)^{\frac{1}{2}}}{(-\Delta)^{\frac{1}{2}}} u_1(x).$$

The α-modulation space estimate for $\cos(t\Delta)u_0(x)$ is equivalent to the α-Modulation space estimate for $e^{it|\Delta|^{\frac{\alpha_o}{2}}}u_0(x)$. Furthermore, to see why $\alpha_o = 1$ corresponds to what is called the (half-) wave equation we can rewrite Wave equation (1.11) as

$$(\partial_t + i|\xi|)(\partial_t - i|\xi|)\hat{u}(t,\xi) = 0,$$

and if we consider the second factor of this factorized form we have

$$(\partial_t - i|\xi|)\hat{u}(t,\xi) = 0.$$

Now working backwards we have

$$\begin{aligned} &(\partial_t - i|\xi|)\hat{u}(t,\xi) = 0 \Rightarrow, \\ &i\partial_t\hat{u}(t,\xi) + |\xi|\hat{u}(t,\xi) = 0 \Rightarrow, \\ &i\partial_t u(t,x) + |\Delta|^{\frac{1}{2}}u(t,x) = 0. \end{aligned} \tag{1.12}$$

The other dispersive equation we will consider in this paper is a generalized Klein-Gordon equation. To familiarize ourselves with this dispersive equation we will first consider the Klein-Gordon equation of the form

$$\begin{cases} \partial_{tt}u(t,x) + (I - \Delta)u(t,x) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \text{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = u_1(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $Iu(t,x) = u(t,x)$ is the identity operator. Again we can find the formal solution of this equation by taking the Fourier transform and writing this equation in the equivalent form

$$\begin{cases} \partial_{tt}\hat{u}(t,\xi) + (1 + |\xi|^2)\hat{u}(t,\xi) = 0, & \text{for } (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), & \text{for } \xi \in \mathbb{R}^n, \\ \partial_t\hat{u}(0,\xi) = \hat{u}_1(\xi), & \text{for } \xi \in \mathbb{R}^n. \end{cases}$$

The general solution has the form of

$$\hat{u}(t,\xi) = c_1 \cos\left(t(1 + |\xi|^2)^{\frac{1}{2}}\right) + c_2 \sin\left(t(1 + |\xi|^2)^{\frac{1}{2}}\right),$$

and using the initial conditions to get the relationships

$$\hat{u}_0(\xi) = c_1 \text{ and } \hat{u}_1(\xi) = c_2\left(1 + |\xi|^2\right)^{\frac{1}{2}},$$

we obtain the particular solution

$$\hat{u}(t,\xi) = \cos\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)\hat{u}_0(\xi) + \frac{\sin\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)}{(1+|\xi|^2)^{\frac{1}{2}}}\hat{u}_1(\xi).$$

Again, taking the Fourier inverse we get

$$u(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\cos\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)\hat{u}_0(\xi)e^{ix\xi}d\xi +$$
$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\sin\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)}{(1+|\xi|^2)^{\frac{1}{2}}}\hat{u}_1(\xi)e^{itx\xi}d\xi.$$

Again we will define these Fourier multipliers as the following operators

$$H_{\cos\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)}u_0(x) = \cos\left(t(I-\Delta^2)^{\frac{1}{2}}\right)u_0(x),$$

and

$$H_{\frac{\sin\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)}{(1+|\xi|^2)^{\frac{1}{2}}}}u_1(x) = \frac{\sin\left(t(I-\Delta^2)^{\frac{1}{2}}\right)}{(I-\Delta^2)^{\frac{1}{2}}}u_1(x).$$

Now we can turn our attention to the generalized Klein-Gordon equation. First, we define the *Bessel Potential* $(I-\Delta)^{\alpha_o}$ [23] by

$$(I-\Delta)^{\alpha_o}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{\alpha_o}\hat{f}(\xi)e^{ix\xi}d\xi. \tag{1.13}$$

Then the *Generalized Klein-Gordon equation of order* α_o by

$$\begin{cases} \partial_{tt}u(t,x) + (I-\Delta)^{\alpha_o}u(t,x) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \text{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = u_1(x), & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{1.14}$$

Using the same techniques as before for with the Klein-Gordon equation we find the formal solution

$$u(t,x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^n}\cos\left(t(1+|\xi|^2)^{\frac{\alpha_o}{2}}\right)\hat{u}_0(\xi)e^{ix\xi}d\xi +$$
$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^n}\frac{\sin\left(t(1+|\xi|^2)^{\frac{\alpha_o}{2}}\right)}{(1+|\xi|^2)^{\frac{\alpha_o}{2}}}\hat{u}_1(\xi)e^{ix\xi}d\xi,$$

using the convention of

$$\cos\left(t(I-\Delta)^{\frac{\alpha_o}{2}}\right)u_0(x),\ \text{and}\ \frac{\sin\left(t(I-\Delta)^{\frac{\alpha_o}{2}}\right)}{(I-\Delta)^{\frac{\alpha_o}{2}}}u_1(x),$$

correspond to the appropriate Fourier Multipliers.

Furthermore, like the treatment of the wave equation, items (1.11) through (1.12) we get a *generalized half Klein-Gordon equation* of the form

$$i\partial_t u(t,x) + (I-\Delta)^{\frac{\alpha_o}{2}}u(t,x) = 0. \tag{1.15}$$

Again, with the same calculations we arrive at a solution of

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{it\left(1+|\xi|^2\right)^{\frac{\alpha_o}{2}}}\hat{u}_0(\xi)d^{ix\xi}d\xi \\ &= H_{e^{it(1+|\xi|^2)^{\frac{\alpha_o}{2}}}}u_0(x). \end{aligned}$$

Furthermore, we will use the following convention of

$$H_{e^{it(1+|\xi|^2)^{\frac{\alpha_o}{2}}}}u_0(x) = e^{it(I-\Delta)^{\frac{\alpha_o}{2}}}u_0(x).$$

We will now close this section by providing in detail some known results of dispersive equations. The first two results are the original Strichartz estimates that can be found in [25]. These are non-endpoint estimates and the first of the two deals with the Schrödinger equation.

Theorem 1.4.1. *([25]) Let $u(x,t)$ be a solution of the inhomogeneous free Schrödinger equation*

$$\begin{cases} i\partial_t u(t,x) + \lambda\Delta u(t,x) = g(t,x), & \textit{for } (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & \textit{for } x\in\mathbb{R}^n, \end{cases}$$

where λ be a nonzero constant, $u_0\in L^2(\mathbb{R}^n)$, and $g\in L^p(\mathbb{R}^{n+1})$ for $p=\dfrac{2(n+2)}{n+4}$, then $u\in L^q(\mathbb{R}^{n+1})$ for $q=\dfrac{2(n+2)}{n}$ and

$$||u||_{L^q(\mathbb{R}^{n+1})} \preceq ||f||_{L^2(\mathbb{R}^n)} + ||g||_{L^q(\mathbb{R}^{n+1})}\,.$$

The second result deals with the Klein-Gordon equation.

Theorem 1.4.2. *([25]) Let $u(t,x)$ be a solution to the Klein-Gordon, where $m > 0$*

$$\begin{cases} -\partial_{tt}u(t,x) + \Delta u(t,x) - m^2 u(t,x) = g(t,x), & \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \text{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = u_1(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $B^{\frac{1}{2}}u_0 \in L^2(\mathbb{R}^n)$, $B^{-\frac{1}{2}}u_1 \in L^2(\mathbb{R}^n)$, where $B = (m^2 - \Delta)^{\frac{1}{2}}$, and $g \in L^q(\mathbb{R}^{n+1})$, then $u \in L^q(\mathbb{R}^{n+1})$ and

$$||u||_{L^q(\mathbb{R}^{n+1})} \preceq \left|\left|B^{\frac{1}{2}}u_0\right|\right|_{L^2(\mathbb{R}^n)} + \left|\left|B^{-\frac{1}{2}}u_1\right|\right|_{L^2(\mathbb{R}^n)} + ||g||_{L^q(\mathbb{R}^{n+1})},$$

with the following restrictions on p and q

$$\begin{aligned} &\text{if } n = 1 \ \frac{2(n+1)}{n+3} \leq p \leq \frac{2(n+2)}{n+4}, \ \frac{2(n+2)}{n} \leq q \leq \frac{2(n+1)}{n-1}, \\ &\text{if } n \geq 2 \ 1 < p \leq \frac{6}{5}, \ 6 \leq q < \infty. \end{aligned}$$

After Strichartz results, the non-endpoint estimates for the wave equation were found by both Lindblad and Sogge [21], and Ginibre and Velo [13].

The last result is one found by Tao and Keel [19]. This finds the endpoint estimates for both the Schrödinger and wave equation. Though the theorem is stated in the context of Hilbert spaces, it is easily applied to these differential equations. First, we need to start with a definition.

Definition 1.4.3. A pair (q,r) that satisfies $q, r \geq 2$, $(q,r,\sigma) \neq (2,\infty,1)$, and

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2},$$

is called *σ-admissible*. A pair (q,r) that satisfies $q, r \geq 2$, $(q,r,\sigma) \neq (2,\infty,1)$, and

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2},$$

is called *sharp σ-admissible.*

Now we can state Tao and Keel [19] result.

Theorem 1.4.4. *([19]) Suppose* $t \in \mathbb{R}_+$ *and let* $U(t) : H \to L^2_x(\mathbb{X})$*, where* $(\mathbb{X}, dx)$ *is a measure space, be an operator defined on a Hilbert space* H*. If* U *obeys the following estimates:*

$$||U(t)f||_{L^2_x(\mathbb{X})} \preceq ||f||_H \text{ for all } f \in H, \text{ and} \tag{1.16}$$

$$||U(s)(U(t))^* g||_{L^\infty_x(\mathbb{X})} \preceq |t-s|^{-\sigma} ||g||_{L^1_x(\mathbb{X})}, \tag{1.17}$$

for some $\sigma > 0, t \neq s$*, and all* $g \in L^1_x(\mathbb{X})$*, then for all sharp* σ*-admissible exponent pairs* (q, r) *and* $(\tilde{q}, \tilde{r})$ U *obeys the following estimates*

$$||U(t)f||_{L^q_t L^r_x} \preceq ||f||_H,$$

$$\left|\left|\int (U(s))^* F(s) ds\right|\right|_H \preceq ||F||_{L^{q'}_t L^{r'}_x},$$

$$\left|\left|\int_{s<t} U(t)(U(s))^* F(s) ds\right|\right|_{L^q_t L^r_x} \preceq ||F||_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}.$$

Furthermore, for all t *and* s*, and* $g \in L^1_x(\mathbb{X})$*, if the equation* (1.17) *in the hypothesis can be strengthen to*

$$||U(s)(U(t))^* g||_{L^\infty_x(\mathbb{X})} \preceq (1 + |t-s|)^{-\sigma} ||g||_{L^1_x(\mathbb{R}^n)},$$

then the conclusion can be extend to all σ*-admissible exponent pairs* (q, r) *and* $(\tilde{q}, \tilde{r})$*.*

1.5 Thesis Outline

The remainder of this thesis is organized as follows: in Chapter 2 we introduce the Besov, Modulation, and α-Modulation spaces. We will discuss the construction of these function spaces, behavior of these function spaces and various embedding properties of these function spaces. We also discuss some recent developments of asymptotic estimates of Fourier Multipliers on these function spaces.

In Chapter 3 we state our main results dealing with asymptotic estimates of several Fourier Multipliers on the α-Modulation space. We will discuss the existing theory in harmonic analysis needed for this dissertation as well as develop some basic estimates. We present the proofs to these main results followed by asymptotic estimates for homogeneous solutions of several well known PDEs on the α-Modulation space.

In Chapter 4 we conclude with presenting several results detailing when a unique local solution exists for some nonlinear PDEs. These results are obtained by using the results from Chapter 3 and using a fixed point theorem.

Chapter 2

Function Spaces

In this chapter we will look at the construction of three well know function spaces: Besov, modulation, and α-modulation. Then we will discuss some properties, embedding and multiplier theorems. First we will give the definition of some basic function spaces that will be used throughout this chapter.

Let $\mathscr{C}(\mathbb{R}^n)$ be the set of all complexed-valued, bounded, and uniformly continuous functions on $\mathbb{R}^n$. Now define $\mathscr{C}^m(\mathbb{R}^n)$ to be the set of all functions, f, such that $\partial^\alpha f \in \mathscr{C}(\mathbb{R}^n)$ for all multi-indices α', where $|\alpha'| \leq m$. See [26] for a full treatment of these function spaces.

For $k \geq 0$ an integer, define the *Sobolev space*, denoted $W^{p,k}(\mathbb{R}^n)$, be the set of all locally integrable functions f on $\mathbb{R}^n$ such that $\partial^{\alpha'} f$ exists in the weak sense and belongs to $L^p(\mathbb{R}^n)$ for all multi-indices $|\alpha'| \leq k$. See [26, 7, 1] for more information about this classic space.

2.1 Besov Spaces

First introduced in 1959 by Besov [4] the Besov space is a function space that measures functions in a different way than the L^p function spaces. The Besov space breaks the spatial domain into different regions using the dyadic decomposition, that is, on regions localized in $\left\{\xi : |\xi| \sim 2^j\right\}$. The construction of the Besov space relies on the Littlewood-Paley decomposition operators, $\triangle j$.

Let $\{\phi_j\}_{j=0}^{\infty}$ be a sequence of functions in $\mathscr{S}(\mathbb{R}^n)$ and $\alpha' = (\alpha_1, \cdots, \alpha_n')$ with

$\alpha_i' \in \mathbb{Z}_*$ for all $1 \le i \le n$, be a multi-index such that the following is satisfied:

$$\begin{cases} \operatorname{supp}\phi_0 \subset \{\xi : |\xi| \le 2\}\,, & \\ \operatorname{supp}\phi_j \subset \left\{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\right\}, & \text{for } j \in \{1,2,\cdots\}\,, \\ 2^{j|\alpha'|}|D^{\alpha'}\phi_j(\xi)| \le c_{\alpha'}, & \text{where } c_{\alpha'} > 0 \text{ is a constant,} \\ \displaystyle\sum_{j=0}^{\infty}\phi_j(\xi) \equiv 1. & \end{cases} \tag{2.1}$$

Denote the set of all functions that satisfy condition (2.1) as X_B. First note X_B is not empty. A standard construction of such a function follows by letting ρ be a smooth radial bump function supported in the ball centered at the origin with radius 2 such that $\rho(\xi) = 1$ if $|\xi| < 1$ and $\rho(\xi) = 0$ if $|\xi| \ge 2$. Now define ϕ to be

$$\phi(\xi) = \rho(\xi) - \rho(2\xi).$$

Define the sequence of functions $\{\phi_j(\xi)\}_{k=0}^{\infty}$ by

$$\begin{cases} \phi_j(\xi) = \phi(2^{-j}\xi), & \text{if } j \in \mathbb{N}, \\ \phi_0(\xi) = 1 - \displaystyle\sum_{j=1}^{\infty}\phi_j(\xi). & \end{cases} \tag{2.2}$$

The sequence of functions defined in (2.2) satisfies the properties in (2.1). For any sequence of functions $\{\phi_j\}_{j=0}^{\infty} \in X_B$, define the Littlewood-Paley decomposition operators, $\triangle j$, by

$$\triangle j = \mathscr{F}^{-1}\phi_j\mathscr{F} \text{ for all } j \in \mathbb{N} \cup \{0\}\,.$$

The Besov norm, denoted by $||f||_{B^s_{p,q}(\mathbb{R}^n)}$, is defined by

$$||f||_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{sjq}\,||\triangle_j f||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}}.$$

Then the Besov space, denoted by $B^s_{p,q}(\mathbb{R}^n)$, is a set of functions $f \in \mathscr{S}'(\mathbb{R}^n)$ where $||f||_{B^s_{p,q}(\mathbb{R}^n)} < \infty$. The reader is directed to [26, 12, 18] for this and other similar constructions of the Besov space.

The Besov space enjoys many properties some of which we will list below. The reader is directed to [26] for a more exhaustive discussion. First we can make a statement as to how Cauchy sequences behave in $B^s_{p,q}(\mathbb{R}^n)$.

Proposition 2.1.1. *([26]) If $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, then $B^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space. If $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, then $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space.*

The construction of the Besov space is independent of the choice of $\{\phi_j\}_{j=0}^{\infty}$. This is formalized in the next proposition.

Proposition 2.1.2. *([26]) If $\{\phi_j\}_{j=0}^{\infty}, \{\varphi_j\}_{j=0}^{\infty} \in X_B(\mathbb{R}^n)$, then they genarate equivalent quasi-norms on $B^s_{p,q}(\mathbb{R}^n)$.*

The Besov space enjoys several embedding properties with some of the other function spaces discussed already.

Proposition 2.1.3. *([26]) The followings embeddings are valid:*

1. $\mathscr{S}(\mathbb{R}^n) \subset B^s_{p,q}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ *for* $0 < p, q \leq \infty, s \in \mathbb{R}$.
2. $\mathscr{S}(\mathbb{R}^n)$ *is dense in* $B^s_{p,q}(\mathbb{R}^n)$.
3. $B^s_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ *for* $1 \leq p < \infty, 0 < q \leq \infty, s > 0$.
4. $B^0_{p,1}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset B^0_{p,\infty}(\mathbb{R}^n)$ *for* $1 \leq p \leq \infty$.
5. $B^0_{\infty,1}(\mathbb{R}^n) \subset \mathscr{C}(\mathbb{R}^n) \subset B^0_{\infty,\infty}(\mathbb{R}^n)$.
6. $B^m_{\infty,1}(\mathbb{R}^n) \subset \mathscr{C}^m(\mathbb{R}^n) \subset B^s_{\infty,\infty}(\mathbb{R}^n)$ *for* $1 \leq p \leq \infty, m = 1, 2, \cdots$.
7. $B^s_{p,q_0} \subset B^s_{p,q_1}$*for* $0 < p \leq \infty, 0 < q_1 \leq q_1 \leq \infty, s \in \mathbb{R}$.
8. $B^s_{p,\min(p,q)} \subset B^s_{p,\max(p,q)}$*for* $0 < p, q \leq \infty, s \in \mathbb{R}$.
9. $B^s_{p,1}(\mathbb{R}^n) \subset W^{p,k}(\mathbb{R}^n) \subset B^{,}_{p,\infty}(\mathbb{R}^n)$ *for* $1 \leq p \leq \infty, m = 1, 2, \cdots$.

Note that $\mathscr{S}(\mathbb{R}^n)$ is not dense in $B^s_{\infty,q}(\mathbb{R}^n)$ or in $B^s_{p,\infty}(\mathbb{R}^n)$. Next we can say something about the diversity of $B^s_{p,q}(\mathbb{R}^n)$.

Proposition 2.1.4. *([26]) For $0 < p_0, p_1, q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$, then*

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^n) = B^{s_1}_{p_1,q_1}(\mathbb{R}^n),$$

if and only if $s_0 = s_1$, $p_0 = p_1$, and $q_0 = q_1$.

This is to say that two Besov spaces coincide if and only if all of the parameters agree.

The next few propositions describe other types of properties. To start, there are certain conditions that will make the Besov space a multiplication algebra.

Proposition 2.1.5. *([18]) For functions $f, g \in B^s_{p,q}(\mathbb{R}^n)$ and $s > \dfrac{n}{p}$, we have*

$$||fg||_{B^s_{p,q}(\mathbb{R}^n)} \preceq ||f||_{B^s_{p,q}(\mathbb{R}^n)} \, ||g||_{B^s_{p,q}(\mathbb{R}^n)} \, .$$

For a function $f \in B^s_{p,q}(\mathbb{R}^n)$, there is a well known scaling property. Recall the definition of δ^a from Chapter 1 Section 1.2 equation (1.3).

Proposition 2.1.6. *([18]) The following estimate is valid*

$$\left|\left|\delta^\lambda f\right|\right|_{B^s_{p,q}(\mathbb{R}^n)} \preceq \lambda^{-\frac{n}{p}} \max\{2, \lambda^s\} \, ||f||_{B^s_{p,q}(\mathbb{R}^n)} \, .$$

The dual space of $B^s_{p,q}(\mathbb{R}^n)$ is well understood. It is formalized as:

Proposition 2.1.7. *([26, 27]) For $0 < p, q < \infty$ and $s \in \mathbb{R}$, the dual space of $B^s_{p,q}(\mathbb{R}^n)$, denoted $(B^s_{p,q}(\mathbb{R}^n))'$, is given by*

$$(B^s_{p,q}(\mathbb{R}^n))' = B^{-s+n\left(\frac{1}{\min\{p,1\}}-1\right)}_{p',q'}.$$

$B^s_{p,q}(\mathbb{R}^n)$ has a few isomorphic properties. The next proposition deals with the Bessel Potential $(I - \Delta)^{\frac{\sigma}{2}}$ which was defined in Chapter 1 Section 1.4 equation (1.13).

Proposition 2.1.8. *([26]) $I_\sigma : B^s_{p,q}(\mathbb{R}^n) \to B^{s-\sigma}_{p,q}(\mathbb{R}^n)$ is an isomorphism for $s, \sigma \in \mathbb{R}$ and $0 < p, q \leq \infty$.*

Recall the definition of τ^y in Chapter 1 Section 1.2, equation (1.2). Note that

$$\begin{aligned} \mathscr{F}^{-1}\phi_j \mathscr{F} \tau^y f(x) &= \mathscr{F}^{-1}\phi_j e^{-iy\xi} \mathscr{F} f \\ &= \tau^y \mathscr{F}^{-1}\phi_j \mathscr{F} f(x) \\ &= \mathscr{F}^{-1}\phi_j \mathscr{F} f(x-y). \end{aligned}$$

Thus $\triangle_j$ commutes with translation. Then we have the following proposition:

Proposition 2.1.9. *([26])* $\tau^y : B^s_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n)$ *is an isomorphism for* $s \in \mathbb{R}^n$ *and* $0 < p, q \leq \infty$.

The next proposition describes a multiplier theorem that is associated with $B^s_{p,q}(\mathbb{R}^n)$. Define the norm $||\cdot||_N$ [26] by

$$||\mu||_N = \sup_{|\alpha'| \leq N} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|^2)^{\frac{|\alpha'|}{2}} |D^{\alpha'} \mu(\xi)|,$$

where α' is a multi-index and N is a positive integer.

Proposition 2.1.10. *([26]) Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$. *If* $\mu \in \mathscr{C}^\infty(\mathbb{R}^n)$, $f \in B^s_{p,q}(\mathbb{R}^n)$, *and if* N *is sufficiently large, then there exists a constant* $c > 0$ *such that*

$$||H_\mu f||_{B^s_{p,q}(\mathbb{R}^n)} \leq c \, ||\mu||_N \, ||f||_{B^s_{p,q}(\mathbb{R}^n)} .$$

Here we see that the Besov norm is preserved.

$B^s_{p,q}(\mathbb{R}^n)$ enjoys a couple of real and complex interpolation properties. Let $\mathscr{H}$ be a linear complex Hausdorff space. A_0, A_1 be quasi-Banach spaces such that $A_0 \subset \mathscr{H}$ and $A_1 \subset \mathscr{H}$ where "$\subset$" means linear and continuous embedding. Define $k(t, a)$ as

$$k(t, a) = \inf_{\substack{a_0 \in A_0 \\ a_1 \in A_1}} \left(||a_0||_{A_1} + t \, ||a_1||_{A_1} \right).$$

This is known as the *k-functional* [3, 26]. $A_0 + A_1$ is the set defined as

$$A_0 + A_1 = \{ a \in \mathscr{A} : a = a_0 + a_1 \text{ where } a_0 \in A_0 \text{ and } a_1 \in A_1 \}.$$

Define the *real interpolation space*, $(A_0, A_1)_{\Theta,q}$, by

$$(A_0, A_1)_{\Theta,q} = \left\{ a \in A_0 + A_1 : ||a||_{(A_0,A_1)_{\Theta,q}} < \infty \right\},$$

where $||a||_{(A_0,A_1)_{\Theta,q}}$ is defined as

$$||a||_{(A_0,A_1)_{\Theta,q}} = \left(\int_0^\infty \left(t^{-\Theta} k(t, a) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

and with the usual modification made when $q = \infty$. For this construction and a more in-depth treatment of real interpolation see [3]. With the above setup we state the following proposition.

Proposition 2.1.11. *([26]) For $0 < \Theta < 1$, $0 < q, q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$ where $s_0 \neq s_1$ and $s = (1-\Theta)s_0 + \Theta s_1$, then for $0 \leq p \leq \infty$*

$$\left(B^{s_0}_{p,q_0}(\mathbb{R}^n), B^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\Theta,q} = B^s_{p,q}(\mathbb{R}^n).$$

Furthermore, if $s_0, s_1 \in \mathbb{R}$ with $s = (1-\Theta)s_0 + \Theta s_1$ and $0 < p_0, p_1 < \infty$ with

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1},$$

then

$$\left(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n)\right)_{\Theta,p} = B^s_{p,p}(\mathbb{R}^n).$$

For complex interpolation, let $A = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and the closure of A, denoted by $\overline{A}$, to be $\overline{A} = \{z \in \mathbb{R} : 0 \leq \mathrm{Re}(z) \leq 1\}$. We say f is $\mathscr{S}'(\mathbb{R}^n)$*-analytic function in A* if it satisfies the following three conditions:

- for fixed $z \in \overline{A}$, $f(z) \in \mathscr{S}'(\mathbb{R}^n)$,
- $\mathscr{F}^{-1}\phi\mathscr{F}f(x,z)$ is a uniformly continuous and bounded function in $\mathbb{R}^n \times \overline{A}$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$,
- $\mathscr{F}^{-1}\phi\mathscr{F}f(x,z)$ is an analytic function in A for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$ and every fixed $x \in \mathbb{R}^n$.

For $f(z)$, an $\mathscr{S}'(\mathbb{R}^n)$-analytic function in A, define $F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))$ where $s_0, s_1 \in \mathbb{R}$ and $0 < p_0, p_1, q_0, q_1 \leq \infty$ to be

$$F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n)) = \Big\{ f : f(it) \in B^{s_0}_{p_0,p_0}(\mathbb{R}^n), f(1+it) \in B^{s_1}_{p_1,p_1}(\mathbb{R}^n) \text{ for all } t \in \mathbb{R}, ||f||_{F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))} < \infty \Big\},$$

and where $||f||_{F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))}$ is defined as

$$||f||_{F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))} = \max_{l=0,1} \sup_{t\in\mathbb{R}} ||f(l+it)||_{B^{s_l}_{p_l,p_l}(\mathbb{R}^n)}.$$

Note $F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))$ is a quasi-Banach space. For $0 < \Theta < 1$ define $(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))_\Theta$ by

$$(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))_\Theta = \Big\{ g : \text{there exists } f \in F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n)) \text{ with } g = f(\Theta) \Big\}.$$

Define the norm, $||\cdot||_{(B^{s_0}_{p_0,p_0}(\mathbb{R}^n),B^{s_1}_{p_1,p_1}(\mathbb{R}^n))_\Theta}$, on this space by

$$||g||_{(B^{s_0}_{p_0,p_0}(\mathbb{R}^n),B^{s_1}_{p_1,p_1}(\mathbb{R}^n))_\Theta} = \inf_{f\in F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n),B^{s_1}_{p_1,p_1}(\mathbb{R}^n))} ||f||_{F(B^{s_0}_{p_0,p_0}(\mathbb{R}^n),B^{s_1}_{p_1,p_1}(\mathbb{R}^n))}. \tag{2.3}$$

See [3] for more details of complex interpolation. With this setup we have the following proposition.

Proposition 2.1.12. *([26]) Let $s_0, s_1 \in \mathbb{R}$ and $0 < p_0, p_1, q_0, q_1 \leq \infty$. If $s, p,$ and q satisfy*

$$s = (1-\Theta)s_0 + \Theta s_1,\ \frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1},$$

then $(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n))_\Theta = B^s_{p,q}(\mathbb{R}^n)$.

A recent result for unimodular Fourier multipliers was developed by G. Zhao, J. Chen, D. Fan, and W. Guo in 2015 [28]. Their result can be summed up as follows

Theorem 2.1.13. *([28]) Let $\beta > 0$, and μ be a real-valued $C^\infty(\mathbb{R}^n\backslash\{0\})$ function which is homogeneous of degree β. Suppose that the Hessian matrix of μ is non-degenerate at $\mathbb{R}^n\backslash\{0\}$. Let $0 \leq p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$ for $i = 1, 2$. Then the Fourier multiplier $e^{i\mu(D)}$ is bounded from $B^{s_1}_{p_1,q_1}(\mathbb{R}^n)$ to $B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ if and only if*

$$\begin{cases} \frac{1}{p_2} \leq \frac{1}{2} \leq \frac{1}{p_1}, \\ s_2 - \frac{n}{p_2} = s_1 - \frac{n}{p_1} + \theta\beta n \min\left\{\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2}\right\}, \\ \left(\frac{1}{q_2} - \frac{1}{q_2}\right)\chi_{\{0,1\}}(\theta) \leq 0, \end{cases} \tag{2.4}$$

holds for some $\theta \in [0, 1]$. Moreover, we have the following asymptotic estimates for the operator norm

$$||e^{i\mu(D)}||_{B^{s_1}_{p_1,q_1}(\mathbb{R}^n)\to B^{s_2}_{p_2,q_2}(\mathbb{R}^n)} \sim \begin{cases} (B(\theta A, (1-\theta)A))^{\frac{1}{q_2}-\frac{1}{q_1}}, & when\ \frac{1}{q_2} > \frac{1}{q_1}, \\ 1, & when\ \frac{1}{q_2} \leq \frac{1}{q_1} \end{cases}$$

where $\theta \in (0, 1)$ and $\theta \in [0, 1]$ respectively if condition (2.4) holds, where $A = n \min\left\{\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2}\right\} \frac{1}{\frac{1}{q_2} - \frac{1}{q_1}}$ and $B(p, q)$ is the beta function defined as

$$B(p, q) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Precisely, for fixed $\frac{1}{p_2} < \frac{1}{2} < \frac{1}{p_1}$ *and* $\frac{1}{q_2} > \frac{1}{q_1}$*, we have the following blow-up rates*

$$||e^{i\mu(D)}||_{B^{s_1}_{p_1,q_1}(\mathbb{R}^n)\to B^{s_2}_{p_2,q_2}(\mathbb{R}^n)} \sim \begin{cases} \theta^{\frac{1}{q_1}-\frac{1}{q_2}}, \; as\ \theta \to 0^+, \\ (1-\theta)^{\frac{1}{q_1}-\frac{1}{q_2}}, \; as\ \theta \to 1^-. \end{cases}$$

2.2 Modulation Spaces

The next function space we will construct will be the modulation space. This function space was first introduced by Feichtinger in 1983 [8]. There are two common constructions of the modulation space: a continuous, and a discrete construction. The continuous definition of the modulation space makes use of the *short-time Fourier transform of a function* f *with respect to* $g \in \mathscr{S}(\mathbb{R}^n)$, denoted by $V_g f(x,\xi)$, which is defined as

$$V_g f(x,\xi) = \int_{\mathbb{R}^n} \overline{g(t-x)} f(t) e^{-it\xi} dt,$$

for $x, \xi \in \mathbb{R}^n$. Here we say g is a *window function*. For $0 < p, q < \infty$ we define the norm $||f||_{\tilde{M}^s_{p,q}(\mathbb{R}^n)}$ by

$$||f||_{\tilde{M}^s_{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_g f(x,\xi)|^p dx \right)^{\frac{q}{p}} (1+|\xi|^2)^{\frac{sq}{2}} d\xi \right)^{\frac{1}{q}},$$

with the usual modifications if p or q are infinite. Then the modulation space, denoted by $M^s_{p,q}(\mathbb{R}^n)$ is defined as the set of all functions $f \in \mathscr{S}'(\mathbb{R}^n)$ where $||f||_{\tilde{M}^s_{p,q}(\mathbb{R}^n)} < \infty$. The reader is directed to [8, 9, 16] for more details about this and similar continuous construction of the modulation space.

To define the modulation space in a discrete way let Q_k be the unit closed cube with center k, $c < 1$, $C > 1$, and $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in C^\infty_c$ be a sequence of functions such that

$$\begin{cases} |\sigma_k(\xi)| \geq c, \text{ for all } \xi \in Q_k, \\ \operatorname{supp} \sigma_k \subset \{\xi : |\xi - k| < C\}, \\ \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1, \text{ for all } \xi \in \mathbb{R}^n, \\ |\partial^{\alpha'} \sigma_k(\xi)| \preceq 1. \end{cases} \tag{2.5}$$

Denote the set of all sequences of functions that satisfy conditions (2.5) by $X_M(\mathbb{R}^n)$. Note that $X_M(\mathbb{R}^n)$ is not empty. To see this let $\rho \in \mathscr{S}(\mathbb{R}^n)$ where $\rho : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function with support on the ball $B(0, \sqrt{n})$ and $\rho(\xi) = 1$ if $|\xi| \leq \frac{\sqrt{n}}{2}$. Define ρ_k by

$$\rho_k(\xi) = \rho(\xi - k),$$

for $k \in \mathbb{Z}^n$. Now define σ_k by

$$\sigma_k(\xi) = \rho_k(\xi) \left(\sum_{j \in \mathbb{Z}^n} \rho_j(\xi) \right)^{-1}.$$

Thus, the sequence of functions $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ defined above satisfy condition (2.5).

For any sequence of functions $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ in $X_M(\mathbb{R}^n)$ and fixed $k \in \mathbb{Z}^n$ define the *smooth projection of f onto $\{\xi : |\xi - k| < C\}$*, denoted by $\square_k f$, as

$$\square_k = \mathscr{F}^{-1} \sigma_k \mathscr{F}.$$

For $0 < p, q \leq \infty$ define the norm $||f||_{M^s_{p,q}(\mathbb{R}^n)}$ by

$$||f||_{M^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \, ||\square_k f||_p^q \right)^{\frac{1}{q}}.$$

Now we can define the modulation space, $M^s_{p,q}(\mathbb{R}^n)$, for all $f \in \mathscr{S}'(\mathbb{R}^n)$ where $||f||_{M^s_{p,q}(\mathbb{R}^n)} < \infty$. Note that $M^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space and when $1 \leq p, q \leq \infty$, then $M^s_{p,q}(\mathbb{R}^n)$ is a Banach space [8, 27]. Furthermore, we have the exact nature of the modulation space $M^s_{p,q}(\mathbb{R}^n)$ when we pick different $\phi \in \mathbb{X}_M$ and a relationship between the discrete and continuous definitions.

Proposition 2.2.1. *([8]) If $\{\sigma_k\}_{k \in \mathbb{Z}^n}, \{\varphi_k\}_{k \in \mathbb{Z}^n} \in X_M(\mathbb{R}^n)$, they generate the same quasi-Banach space $M^s_{p,q}(\mathbb{R}^n)$.*

Proposition 2.2.2. *([8]) For $0 < p, q \leq \infty$, $s \in \mathbb{R}$, then $||\cdot||_{\tilde{M}^s_{p,q}(\mathbb{R}^n)}$ and $||\cdot||_{M^s_{p,q}(\mathbb{R}^n)}$ are equivalent norms on the modulation space $M^s_{p,q}(\mathbb{R}^n)$.*

We will see later that $M^s_{p,q}(\mathbb{R}^n) = M^{s;0}_{p,q}(\mathbb{R}^n)$ and other properties of the modulation space $M^s_{p,q}(\mathbb{R}^n)$ will be an immediate consequence of the properties for the

α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. Thus, we will forgo all other properties of the modulation space $M^s_{p,q}(\mathbb{R}^n)$. Instead we will cover more recent results of the modulation space $M^s_{p,q}(\mathbb{R}^n)$, as it relates to the study of PDEs and Fourier multipliers. For convenience let $M^0_{p,q}(\mathbb{R}^n) = M_{p,q}(\mathbb{R}^n)$.

Bényi, Gröchenig, Okoudjou, and Roders [2] showed the following multiplier theorem:

Theorem 2.2.3. *([2]) If $\alpha_o \in [0,2]$, then the Fourier multiplier with symbol $e^{i|\xi|^{\alpha_o}}$ is bounded from $M_{p,q}(\mathbb{R}^n)$ into $M_{p,q}(\mathbb{R}^n)$ for all $1 \le p,q \le \infty$ and for any dimension $n \ge 1$.*

As of consequence of Theorem 2.2.3 A. Bényi, K. Gröchenig, K. Okoudjou, and L. Roders were able to establish the following Fourier Multiplier Theorem:

Corollary 2.2.4. *([2]) Let $u(t,x)$ be the solution of the Cauchy problem of dispersive equation (1.9) that takes the form of equation (1.10). For $\alpha_o = 2$, $t > 0$, and any dimension, then*

$$||u||_{M_{p,q}(\mathbb{R}^n)} = \left|\left|e^{it\Delta}u_0\right|\right|_{M_{p,q}(\mathbb{R}^n)} \preceq (1+t)^{\frac{n}{2}} ||u_0||_{M_{p,q}(\mathbb{R}^n)} .$$

Wang and Hudzik [27] used an almost orthogonality argument and simple calculations to show the following multiplier theorem:

Theorem 2.2.5. *([27]) Let $s \in \mathbb{R}$, $2 \le p \le \infty$, and $0 \le q \le \infty$, then*

$$\left|\left|e^{it\Delta}f\right|\right|_{M^s_{p,q}(\mathbb{R}^n)} \preceq (1+t)^{-n\left(\frac{1}{2}-\frac{1}{p}\right)} ||f||_{M^s_{p',q}(\mathbb{R}^n)} .$$

Chen, Fan, and Sun [5] also use an almost orthogonality argument and techniques for oscillating integrals to show:

Theorem 2.2.6. *([5]) If $\alpha_o = 1$, $t > 1$, $s \in \mathbb{R}$, and $1 \le p,q \le \infty$, then*

$$\left|\left|e^{it|\Delta|^{\frac{\alpha_o}{2}}}f\right|\right|_{M^s_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{2}-\frac{1}{p}\right|} ||f||_{M^s_{p,q}(\mathbb{R}^n)} ,$$

and:

Theorem 2.2.7. *([5]) If $t > 1$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, and $\alpha_o > \frac{1}{2}$ but $\alpha_o \neq 1$, then*

$$\left|\left|e^{it|\Delta|^{\frac{\alpha_o}{2}}} f\right|\right|_{M^s_{p,q}(\mathbb{R}^n)} \preceq ||f||_{M^s_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{2}-\frac{1}{p}\right|} ||f||_{M^{s+\gamma(\alpha_o)}_{p,q}(\mathbb{R}^n)},$$

where

$$\gamma(\alpha_o) = (\alpha_o - 2)n \left|\frac{1}{2} - \frac{1}{p}\right|.$$

2.3 α-Modulation Spaces

The final function space we will look at is the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. Much like the modulation space $M^s_{p,q}(\mathbb{R}^n)$, there are two ways of constructing it. One way is the continuous definition using an admissible covering, and the other way is the discrete construction.

Let $\#$ denote the cardinality of a finite set. Suppose $\mathscr{I}$ is a set of countable intervals $I \subset \mathbb{R}^n$ denoted by $\mathscr{I}$ is called *admissible covering of* $\mathbb{R}^n$ if the following is satisfied:

1. $\mathbb{R}^n = \bigcup_{I \in \mathscr{I}} I$,
2. $\#\{I \in \mathscr{I} : x \in I\} \leq m_0$ for all $x \in \mathbb{R}^n$ and where m_0 is some positive integer.

Also, if there exists a constant $0 \leq \alpha \leq 1$ such that $(1 + |\xi|)^\alpha \preceq |I| \preceq (1 + |\xi|)^\alpha$ for all $I \in \mathscr{I}_\alpha$ and for all $\xi \in \mathbb{R}^n$, then $\mathscr{I}_\alpha$ is called *α-covering*. For the sake of convenience, we can let $m_0 = 2$. See [10] for a more detailed treatment of admissible coverings.

Without loss of generality, we can construct a *bounded admissible partition of the unity*, $\{\psi^\alpha_I\}_{I \in \mathscr{I}_\alpha} \in \mathscr{S}(\mathbb{R}^n)$, that is associated with an admissible α-covering. Let $||f||_{\mathscr{F}L^1(\mathbb{R}^n)}$ be defined by

$$||f||_{\mathscr{F}L^1(\mathbb{R}^n)} = \left|\left|\mathscr{F}^{-1}(f)(\xi)\right|\right|_{L^1(\mathbb{R}^n)},$$

then $\{\psi^\alpha_I\}_{I \in \mathscr{I}_\alpha} \in \mathscr{S}(\mathbb{R}^n)$ satisfies the following:

1. $\sup_{I \in \mathscr{I}_\alpha} ||\psi^\alpha_I||_{\mathscr{F}L^1(\mathbb{R}^n)} < \infty$,

2. $\operatorname{supp} \psi_I^\alpha \subset I$ for all $I \in \mathscr{I}_\alpha$,

3. $\sum_{I \in \mathbb{I}_\alpha} \psi_I^\alpha(\xi) \equiv 1$ for all $\xi \in \mathbb{R}^n$.

Finally, define the *segmentation operator*, denoted by $\mathscr{P}_I^\alpha$, by

$$\mathscr{P}_I^\alpha(f) = \mathscr{F}^{-1}\psi_I^\alpha \mathscr{F}(f),$$

for all $I \in \mathscr{I}_\alpha$ and for all $f \in \mathscr{S}'(\mathbb{R}^n)$.

Let $1 \leq p, q < \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$. Suppose $\mathscr{I}_\alpha$ is an admissible α-covering of $\mathbb{R}^n$ and $\{\psi_I^\alpha\}_{I \in \mathscr{I}_\alpha}$ is the associated bounded admissible partition of the unity. Define the *α-modulation space*, denoted by $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$, on the set of tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ that satisfies:

$$||f||_{\tilde{M}_{p,q}^{s,\alpha}(\mathbb{R}^n)} \alpha = \left(\sum_{I \in \mathscr{I}_\alpha} ||\mathscr{P}_I^\alpha(f)||_{L^p(\mathbb{R}^n)}^q \left(1 + |\omega_I|\right)^{sq} \right)^{\frac{1}{q}} < \infty,$$

where $\omega_I \in I$ for all $I \in \mathscr{I}_\alpha$, and making the usual modification when p and q are infinite.

First note that the definition of the α-modulation space does not depend on the choice of $\{\omega_I\}_{I \in \mathscr{I}}$. See [15, 11] for more details of this construction.

For the discrete construction, let $c < 1$ and $C > 1$ be two positive numbers which relate to the space dimension n, and $0 \leq \alpha < 1$. Suppose $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$ be a sequence of Schwartz functions that satisfies the following:

$$\begin{cases} |\eta_k^\alpha(\xi)| \succeq 1, \text{ if } \left|\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k\right| < c \langle k \rangle^{\frac{\alpha}{1-\alpha}}, \\ \operatorname{supp} \eta_k^\alpha \subset \left\{ \xi \in \mathbb{R}^n : \left|\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k\right| < C \langle k \rangle^{\frac{\alpha}{1-\alpha}} \right\}, \\ \sum_{k \in \mathbb{Z}^n} \eta_k^\alpha(\xi) \equiv 1, \text{ for all } \xi \in \mathbb{R}^n, \\ \langle k \rangle^{\frac{\alpha|\delta|}{1-\alpha}} \left|D^\delta \eta_k^\alpha(\xi)\right| \preceq 1, \text{ for all } \xi \in \mathbb{R}^n \text{ and all multi-index } \delta. \end{cases} \tag{2.6}$$

Denote the set of sequences of functions that satisfies (2.6) by X_A. Note that X_A is not empty. To see this, let ρ be a smooth radial bump function supported on the open ball of radius 2 centered at the origin that satisfies $\rho(\xi) = 1$ when $|\xi| < 1$ and $\rho(\xi) = 0$ when $|\xi| \geq 2$. For any $k \in \mathbb{Z}^n$ define ρ_k^α by

$$\rho_k^\alpha(\xi) = \rho\left(\frac{\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k}{C\langle k\rangle^{\frac{\alpha}{1-\alpha}}}\right).$$

Now define η_k^α by

$$\eta_k^\alpha(\xi) = \rho(\xi)\left(\sum_{l\in\mathbb{Z}^n} \rho_l^\alpha(\xi)\right)^{-1}.$$

Thus this η_k^α satisfies condition (2.6).

For $\{\eta_k^\alpha\}_{k=0}^\infty \in X_A$ define $\square_k^\alpha$ by

$$\square_k^\alpha = \mathscr{F}^{-1}\eta_k^\alpha\mathscr{F}. \tag{2.7}$$

For $0 < p, q \leq \infty$, $s \in \mathbb{R}$, and $\alpha \in [0, 1)$ define the norm $||\cdot||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)}$ by

$$||f||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\square_k^\alpha f||_{L^p(\mathbb{R}^n)}^q\right)^{\frac{1}{q}}. \tag{2.8}$$

We now define the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ as the set of all $f \in \mathscr{S}'$, such that $||f||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} < \infty$. When $\alpha = 1$ it will be understood that we are using the continuous definition of $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.

We will now cover some properties of the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$, and by doing so, add to our understanding of the modulation Space $M_{p,q}^s(\mathbb{R}^n)$ as well. The next few theorems are similar to the theorems we had for the Besov Space $B_{p,q}^\alpha(\mathbb{R}^n)$.

Proposition 2.3.1. *([15]) For $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $\alpha \in [0, 1)$, then $||\cdot||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)}$ and $||\cdot||_{\tilde{M}_{p,q}^{s,\alpha}(\mathbb{R}^n)}$ are equivalent norms on the modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.*

Proposition 2.3.2. *([18]) Let $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ and $\{\tilde{\eta}_k^\alpha\}_{k\in\mathbb{Z}^n}$ be in X_A, then they generate equivalent quasi-norms on $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.*

Proposition 2.3.3. *([18]) For $0 < p, q \leq \infty$, $s \in \mathbb{R}$, and $\alpha \in [0, 1]$, then $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ is a quasi-norm. If $1 \leq p, q \leq \infty$, then $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ is a Banach space.*

Proposition 2.3.4. *([18]) The following embedding is true*

$$\mathscr{S}(\mathbb{R}^n) \subset M_{p,q}^{s,\alpha}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n).$$

If $0 < p, q < \infty$, then $\mathscr{S}(\mathbb{R}^n)$ is dense in $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.

There are conditions which we have for an embedding from $M_{p_1,q_1}^{s_1,\alpha}(\mathbb{R}^n)$ into $M_{p_2,q_2}^{s_2,\alpha}(\mathbb{R}^n)$. These results are summed up as follows:

Proposition 2.3.5. *([18]) Let $0 < p_1 \leq p_2 \leq \infty$ and $0 < q_1, q_2 \leq \infty$. If $q_1 \leq q_2$ and $s_1 \geq s_2 + n\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then*

$$M_{p_1,q_1}^{s_1,\alpha}(\mathbb{R}^n) \subset M_{p_2,q_2}^{s_2,\alpha}(\mathbb{R}^n).$$

If $q_1 > q_2$ and $s_1 > s_2 + n\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + n(1-\alpha)\left(\frac{1}{q_2} - \frac{1}{q_1}\right)$, then

$$M_{p_1,q_1}^{s_1,\alpha}(\mathbb{R}^n) \subset M_{p_2,q_2}^{s_2,\alpha}(\mathbb{R}^n).$$

With this proposition we can say something about the modulation space $M_{p,q}^{s}(\mathbb{R}^n)$, which takes the form of the following corollary, shown by Wang and Hudzik [27].

Corollary 2.3.6. *([27]) If $s_1 \geq s_2$, $0 < p_1 \leq p_2 \leq \infty$, and $0 < q_1 \leq q_2 \leq \infty$, then*

$$M_{p_1,q_1}^{s_1}(\mathbb{R}^n) \subset M_{p_2,q_2}^{s_2}(\mathbb{R}^n).$$

Furthermore, if $q_1 > q_2$ and with $s_1 - s_2 > n\left(\frac{1}{q_2} - \frac{1}{q_1}\right)$, then

$$M_{p,q_1}^{s_1}(\mathbb{R}^n) \subset M_{p,q_2}^{s_2}(\mathbb{R}^n).$$

There are also conditions that guarantee an embedding from $M_{p,q}^{s_1,\alpha_1}(\mathbb{R}^n)$ into $M_{p,q}^{s_2,\alpha_2}(\mathbb{R}^n)$ and dilation property. To be able to state these results first we need to decompose $\mathbb{R}_+^2$ into suitable regions. These decompositions can be found in [18]. Let $0 < p, q \leq \infty$ and $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Now define $R(p, q; \alpha_1, \alpha_2)$ as:

$$R(p, q; \alpha_1, \alpha_2) = \max\left\{0, n(\alpha_1 - \alpha_2)\left(\frac{1}{q} - \frac{1}{p}\right), n(\alpha_1 - \alpha_2)\left(\frac{1}{p} + \frac{1}{p} - 1\right)\right\}. \quad (2.9)$$

Now we decompose $\mathbb{R}^2_+$ into three regions in two different ways. These decompositions will help us understand and simplify $R(p,q;\alpha_1,\alpha_2)$. The first decomposition is defined as $\mathbb{R}^2_+ = S_1 \cup S_2 \cup S_3$ where S_1, S_2, and S_3 are defined as

$$\begin{cases} S_1 = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_+ : \frac{1}{q} \geq \frac{1}{p}, \frac{1}{p} \leq \frac{1}{2} \right\}, \\ S_2 = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_+ : \frac{1}{p} + \frac{1}{q} \geq 1, \frac{1}{p} > \frac{1}{2} \right\}, \\ S_3 = \mathbb{R}^2_+ / \left\{ S_1 \cup S_2 \right\}. \end{cases}$$

See Figure 2.1 for how $\mathbb{R}^2_+$ is decomposed in this case. The second decomposition is defined as $\mathbb{R}^2_+ = T_1 \cup T_2 \cup T_3$ where T_1, T_2, and T_3 are defined as:

$$\begin{cases} T_1 = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_+ : \frac{1}{p} \geq \frac{1}{q}, \frac{1}{p} > \frac{1}{2} \right\}, \\ T_2 = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_+ : \frac{1}{p} + \frac{1}{q} \leq 1, \frac{1}{p} \leq \frac{1}{2} \right\}, \\ T_3 = \mathbb{R}^2_+ / \left\{ S_1 \cup S_2 \right\}. \end{cases}$$

See Figure 2.2 for how $\mathbb{R}^2_+$ is decomposed in this case.

With the above decomposition for $\mathbb{R}^2_+$ we have the following for $R(p,q;\alpha_1,\alpha_2)$. For $\alpha_1 \geq \alpha_2$, we have

$$R(p,q;\alpha_1,\alpha_2) = \begin{cases} n(\alpha_1 - \alpha_2)\left(\frac{1}{q} - \frac{1}{p}\right), & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in S_1, \\ n(\alpha_1 - \alpha_2)\left(\frac{1}{p} + \frac{1}{q} + 1\right), & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in S_2, \\ 0, & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in S_3. \end{cases}$$

If $\alpha_1 < \alpha_2$, then we have

$$R(p,q;\alpha_1,\alpha_2) = \begin{cases} n(\alpha_1 - \alpha_2)\left(\frac{1}{q} - \frac{1}{p}\right), & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in T_1, \\ n(\alpha_1 - \alpha_2)\left(\frac{1}{p} + \frac{1}{q} - 1\right), & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in T_2, \\ 0, & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in T_3. \end{cases}$$

With this understanding of $R(p,q;\alpha_1,\alpha_2)$ we can state under what condition $M^{s_1,\alpha_1}_{p,q}(\mathbb{R}^n)$ is embedded into $M^{s_2,\alpha_2}_{p,q}(\mathbb{R}^n)$.

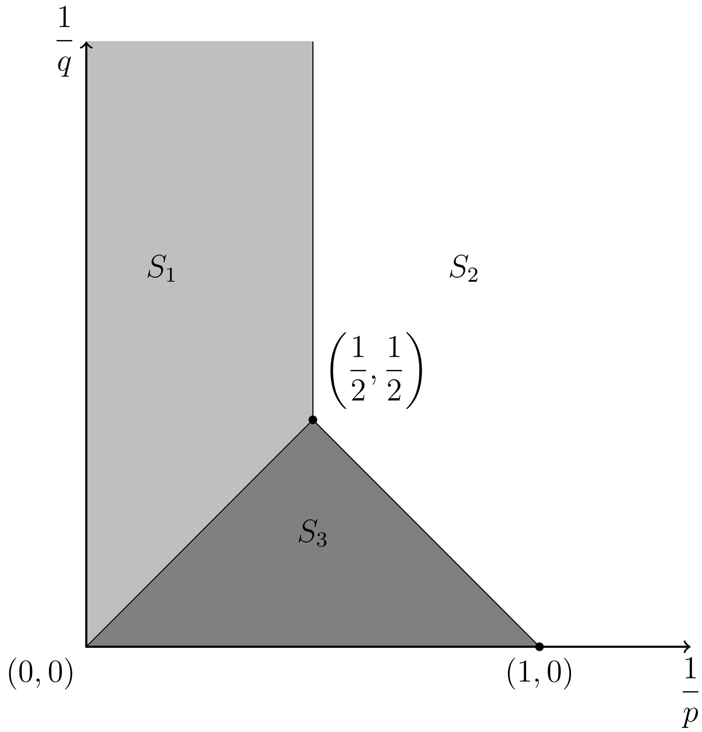

Figure 2.1: Distribution of s_c when $\lambda > 1$.

Proposition 2.3.7. *([18]) For* $(\alpha_1, \alpha_2) \in [0,1) \times [0,1)$*, then*

$$M_{p,q}^{s_1,\alpha_1}(\mathbb{R}^n) \subset M_{p,q}^{s_2,\alpha_2}(\mathbb{R}^n),$$

if and only if $s_1 \geq s_2 + R(p,q;\alpha_1,\alpha_2)$*.*

Now define s_c and s_p as

$$s_c = \begin{cases} R(p,q;1,\alpha), & \lambda > 1, \\ R(p,q;\alpha,1), & \lambda \leq 1, \end{cases}$$

and

$$s_p = n\left(\frac{1}{\min(1,p)} - 1\right).$$

Proposition 2.3.8. *([18]) Let* $0 \leq \alpha < 1$*,* $\lambda > 0$*, and* $s + s_c \neq 0$*, then*

$$\left|\left|\delta^\lambda f\right|\right|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq \lambda^{-\frac{n}{p}} \left(\max\left\{\max\left\{1,\lambda\right\}^{s_p}, \lambda^{s+s_c}\right\}\right) ||f||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)},$$

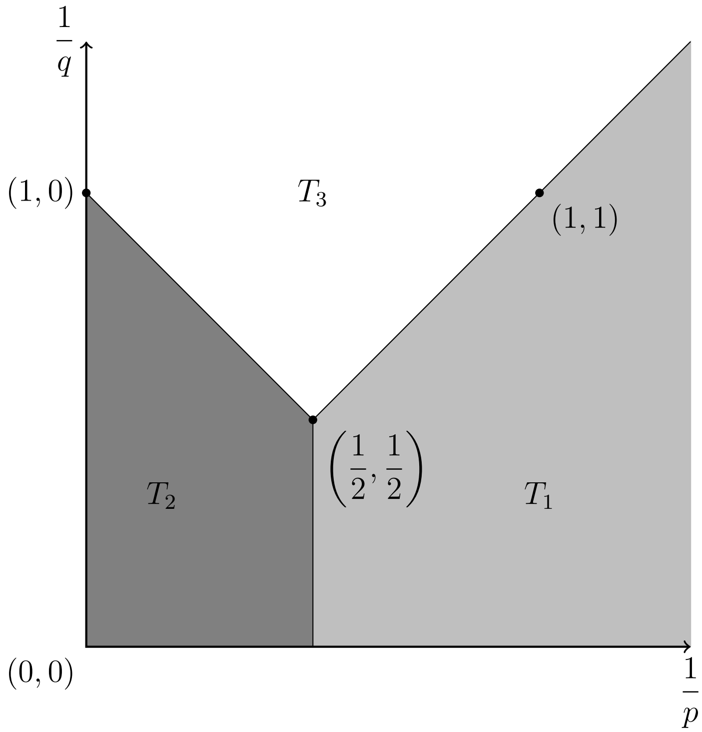

Figure 2.2: Distribution of s_c when $\lambda \leq 1$.

for all $f \in M^{s,\alpha}_{p,q}(\mathbb{R}^n)$. *Conversely, if*

$$\left|\left|\delta^\lambda f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq \lambda^{-\frac{n}{p}} F(\lambda) \left|\left|f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)},$$

holds for some $F : (0,\infty) \to (0,\infty)$ *and* $f \in M^{s,\alpha}_{p,q}(\mathbb{R}^n)$, *then* $F(\lambda) \succeq \max\left\{\max\left\{1,\lambda\right\}^{s_p}, \lambda^{s+s_c}\right\}$.

Furthermore letting $s = -s_c$, then it follows that

$$\left|\left|\delta^\lambda f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq \lambda^{-\frac{n}{p}} F(\lambda) \left|\left|f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)},$$

where F is the following:

$$F(\lambda) = \begin{cases} (\ln(\lambda))^{\max\left\{0,\frac{1}{q}-\frac{1}{p},\frac{1}{q}+\frac{1}{p}-1\right\}}, & \text{when } \lambda > 1, p \geq 1, \\ \lambda^{n\left(\frac{1}{p}-1\right)} (\ln(\lambda))^{\frac{1}{q}}, & \text{when } \lambda > 1, p \leq 1, \\ (\ln(\lambda))^{\max\left\{0,\frac{1}{p}-\frac{1}{q},1-\frac{1}{p}-\frac{1}{q}\right\}}, & \text{when } \lambda \leq 1. \end{cases}$$

The next theorem gives us an understanding of the dual space of the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$.

Proposition 2.3.9. *([18]) For $0 < p, q < \infty$, $s \in \mathbb{R}$, and $\alpha \in [0,1]$, then:*

$$\left(M^{s,\alpha}_{p,q}(\mathbb{R}^n)\right)' = M^{-s+n\alpha\left(\frac{1}{\min(1,p)}-1\right)}_{\max(1,p)',\max(1,q)'}(\mathbb{R}^n).$$

With this proposition, a similar statement can be made about the dual space of the modulation space $M^s_{p,q}(\mathbb{R}^n)$, which can be found in Wang and Hudzik [27] and Han and Wang [18].

Proposition 2.3.10. *([27, 18] For $s \in \mathbb{R}$ and $0 < p, q < \infty$, we have*

$$\left(M^{s}_{p,q}(\mathbb{R}^n)\right)' = M^{-s}_{\max(1,p)',\max(1,q)'}(\mathbb{R}^n).$$

Proposition 2.3.11. *([18]) For $s, \sigma \in \mathbb{R}$ the mapping $(I-\Delta)^{\frac{\sigma}{2}} : M^{s,\alpha}_{p,q}(\mathbb{R}^n) \to M^{s-\sigma,\alpha}_{p,q}(\mathbb{R}^n)$ is an isomorphism.*

This leads us straight to the following corollary:

Corollary 2.3.12. *([27]) For $s, \sigma \in \mathbb{R}$ the mapping $(I-\Delta)^{\frac{\sigma}{2}} : M^{s}_{p,q}(\mathbb{R}^n) \to M^{s-\sigma}_{p,q}(\mathbb{R}^n)$ is an isomorphism.*

The next theorem deals with the understanding of the α-modulation space as an multiplication algebra, i.e. when

$$||fg||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq ||f||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \, ||g||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \, .$$

First we must decompose $\mathbb{R}^2_+$ into two regions, this decomposition can be found in [18]. Define D_1 as

$$D_1 = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_+ : \frac{1}{q} \geq \frac{2}{p}, \frac{1}{p} \leq \frac{1}{2} \right\},$$

and define D_2 as

$$D_2 = \mathbb{R}^2_+ / D_1.$$

Define s_0 as

$$s_0 = \frac{n\alpha}{p} + n(1-\alpha)\left(1 - \min\left\{1, \frac{1}{q}\right\}\right) + \frac{n\alpha(1-\alpha)}{2-\alpha}\left(\frac{1}{q} - \frac{2}{p}\right)$$

when $\left(\frac{1}{p}, \frac{1}{q}\right) \in D_1$ and

$$s_0 = \frac{n\alpha}{p} + n(1-\alpha)\left(\max\left\{1, \frac{1}{p}, \frac{1}{q}\right\} - \frac{1}{q}\right) + \frac{n\alpha(1-\alpha)}{2-\alpha}\left(\max\left\{1, \frac{1}{p}, \frac{1}{q}\right\} - 1\right),$$

when $\left(\frac{1}{p}, \frac{1}{q}\right) \in D_2$. See Figure 2.3 to see such a distribution of s_0. Now we are

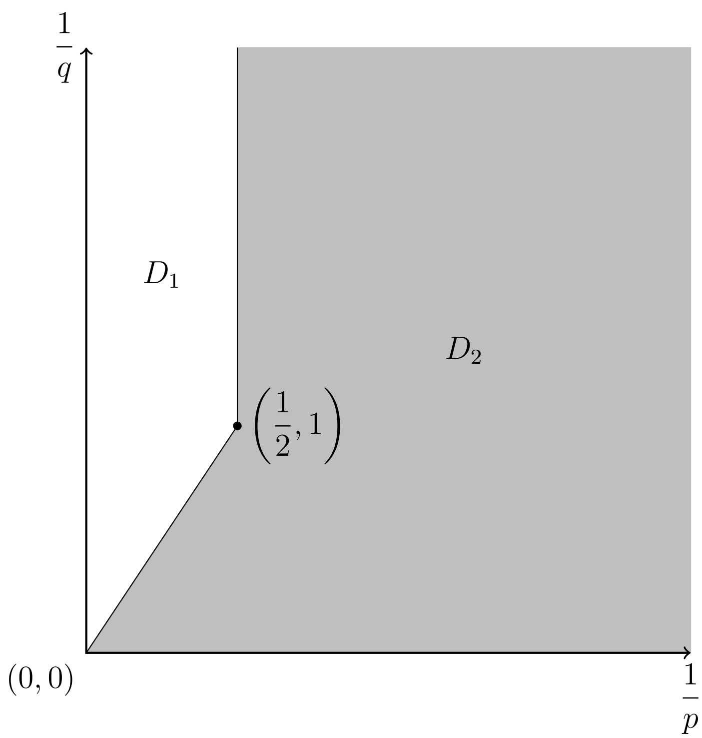

Figure 2.3: Distribution of s_0 for all p and q.

able to state the multiplication algebra theorem.

Proposition 2.3.13. *([18]) If $s > s_0$, then for all $f, g \in M^{s,\alpha}_{p,q}(\mathbb{R}^n)$, then*

$$||fg||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq ||f||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \, ||g||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \,,$$

i.e. the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ is a multiplication algebra.

Using this result we obtain a similar version for the modulation space $M^s_{p,q}(\mathbb{R}^n)$.

Corollary 2.3.14. *([18]) Suppose that*

$$s > \begin{cases} n\left(1 - 1\min\left\{1, \frac{1}{q}\right\}\right), & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in D_1, \\ n\left(\max\left\{1, \frac{1}{p}, \frac{1}{q}\right\} - \frac{1}{q}\right), & \text{when } \left(\frac{1}{p}, \frac{1}{q}\right) \in D_2, \end{cases}$$

then the modulation space $M^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra, i.e.,

$$||fg||_{M^s_{p,q}(\mathbb{R}^n)} \preceq ||f||_{M^s_{p,q}(\mathbb{R}^n)} \, ||g||_{M^s_{p,q}(\mathbb{R}^n)} \, ,$$

for all $f, g \in M^s_{p,q}(\mathbb{R}^n)$.

There are known results for the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ when it comes to complex interpolation. Recall the definition for complex interpolation spaces, equation (2.3) in Section 2.1. We then have the following result:

Proposition 2.3.15. *([18]) For $0 \leq \alpha < 1$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $s = (1-\theta)s_0 + \theta s_1$, then*

$$\left(M^{s_0,\alpha}_{p_0,q_0}(\mathbb{R}^n), M^{s_1,\alpha}_{p_1,q_1}(\mathbb{R}^n)\right)_\theta = M^{s,\alpha}_{p,q}(\mathbb{R}^n).$$

Next we will cover some of the recent results. First, Guo and Chen [17] developed Strichartz estimates for the nonlinear Cauchy problem for dispersive equation (1.9) and the nonlinear Wave equation (1.11) using a TT^* and duality argument. The first result is for the nonlinear Cauchy problem for dispersive equation.

Theorem 2.3.16. *([17]) Suppose $s \in \mathbb{R}$, $q \geq 1$, $\alpha_o \in (0, 2]$ and $\beta \neq 1$, (r, q) and $(\tilde{p}, \tilde{q})$ satisfies*

$$\frac{1}{r} + \frac{n}{2p} \leq \frac{n}{4}, \text{ and } \frac{1}{\tilde{r}} + \frac{n}{2\tilde{p}} \leq \frac{n}{4},$$

with $(r,q,n),(\tilde{p},\tilde{q},\tilde{n}) \neq (2,\infty,2)$, *then*

$$\left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \left(\int_{\mathbb{R}} ||u(t,\cdot)||^r_{L^p_x}\, dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \preceq$$
$$||u_0||_{M^{s+\delta(r,p),\alpha}_{2,q}(\mathbb{R}^n)}$$
$$+\left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{(s+\delta(r,p)+\delta(\tilde{r},\tilde{p}))q}{1-\alpha}} \left(\int_{\mathbb{R}} ||F(t,\cdot)||^{\tilde{r}'}_{L^{\tilde{p}'}_x}\, dt\right)^{\frac{q}{\tilde{r}'}}\right)^{\frac{1}{q}},$$

where $\delta(r,p) = \alpha\left(\frac{n}{2}-\frac{2}{r}-\frac{n}{p}\right)+(2-\beta)\frac{1}{r}$ *and* $F(t,x)$ *is a nonlinear term. More precisely we have:*

$$\left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \left(\int_{\mathbb{R}} \left|\left|\square^k_\alpha e^{it(-\Delta)^{\frac{\alpha_o}{2}}} u_0\right|\right|^r_{L^p(\mathbb{R}^n)} dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \preceq ||u_0||_{M^{s+\delta(r,q),\alpha}_{2,q}(\mathbb{R}^n)},$$

and

$$\left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \left(\left|\left|\int_{\mathbb{R}} e^{i(t-s)(-\Delta)^{\frac{\alpha_o}{2}}} F(s)ds\right|\right|^r_{L^p(\mathbb{R}^n)} dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}$$
$$\preceq \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{(s+\delta(r,p)+\delta(\tilde{r},\tilde{p}))q}{1-\alpha}} \left(\int_{\mathbb{R}} ||F||^{\tilde{r}'}_{L^{\tilde{p}'}(\mathbb{R}^n)}\, dt\right)^{\frac{q}{\tilde{r}'}}\right)^{\frac{1}{q}}.$$

Along similar lines as Theorem 2.3.16, Guo and Chen [17] found Strichartz estimates for the nonlinear Wave equation, which are as follows:

Theorem 2.3.17. *([17]) Let* $s\in\mathbb{R}$, $q\geq 1$, $0\leq\alpha<1$, *and* (p,r) *and* $(\tilde{p},\tilde{r})$ *both satisfy*

$$\frac{n}{2}-\frac{n}{p}-\frac{1}{r}-1>0,$$

and

$$n-1-\frac{n}{p}-\frac{1}{r}-\frac{n}{\tilde{p}}-\frac{1}{\tilde{r}}>0,$$

then the solutions to the wave equation satisfy the following estimate:

$$\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{sq}{1-\alpha}}\left(\int_{\mathbb{R}}||\Box_k^\alpha u||^r_{L^q_x(\mathbb{R}^n)}\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \preceq ||u_0||_{M^{s+\theta(r,p),\alpha}_{2,q}(\mathbb{R}^n)}$$
$$+ ||u_1||_{M^{s+\theta(r,p)-1,\alpha}_{2,q}(\mathbb{R}^n)}$$
$$+ \left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{(s+\theta(r,q)+\theta(\tilde{r},\tilde{p})-1)q}{1-\alpha}}\left(\int_{\mathbb{R}}||\Box_k^\alpha F||^{\tilde{r}'}_{L^{\tilde{p}'}_x(\mathbb{R}^n)}\,dt\right)^{\frac{q}{\tilde{r}'}}\right)^{\frac{1}{q}},$$

where θ is defined as

$$\theta(r,p)=\alpha\frac{n}{n-1}\left(\frac{n-1}{2}-\frac{2}{r}-\frac{n-1}{p}\right)+\frac{n+1}{r(n-1)}.$$

More precisely we have the following three estimates:

$$\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{sq}{1-\alpha}}\left(\int_{\mathbb{R}}\left|\left|\cos(t(-\Delta)^{\frac{1}{2}})u_0\right|\right|^r_{L^q_x(\mathbb{R}^n)}dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \preceq ||u_0||_{M^{s+\theta(r,q),\alpha}_{2,q}(\mathbb{R}^n)},$$

$$\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{sq}{1-\alpha}}\left(\int_{\mathbb{R}}\left|\left|\frac{\sin(t(-\Delta)^{\frac{1}{2}}}{(-\Delta)^{\frac{1}{2}}}u_1\right|\right|^r_{L^q_x(\mathbb{R}^n)}dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \preceq ||u_1||_{M^{s+\theta(r,q)-1,\alpha}_{2,q}(\mathbb{R}^n)},$$

and

$$\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{sq}{1-\alpha}}\left(\int_{\mathbb{R}}\left|\left|\int_{\mathbb{R}}\frac{\sin((t-s)(-\Delta)^{\frac{1}{2}}}{(-\Delta)^{\frac{1}{2}}}F(s)ds\right|\right|^r_{L^q_x(\mathbb{R}^n)}dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \preceq$$
$$\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{(s+\theta(r,q)+\theta(\tilde{r},\tilde{q})-1)q}{1-\alpha}}\left(\int_{\mathbb{R}}||F||^{\tilde{r}'}_{L^{\tilde{p}'}_x(\mathbb{R}^n)}dt\right)^{\frac{q}{\tilde{r}'}}\right)^{\frac{1}{q}}$$

Guo and Chen also found similar estimates for the nonelliptic Schrödinger equation and the Cauchy problem for Wave equation. Also, this result is in line with the results that Keel and Tao found in [19] that extended the results that Strichartz first developed in [25].

Furthermore, Zhao, Chen, and Guo [29] used a direct calculation to find the following estimates for the Fourier multiplier with symbol $e^{i\mu(\xi)}$ where $\mu(\xi)$ is a real-valued function:

Theorem 2.3.18. *([29]) Let $\delta > 0$, $L \in \mathbb{N}$, $L \geq \left[\frac{n}{2}\right] + 1$, $\beta > 0$ and*

$$\mathcal{S}_p(\beta) = \left(\frac{1}{p} - \frac{1}{2}\right) \max\left\{(\beta - 2)n + 2\alpha n, 0\right\}.$$

Assume that μ is a class $C^N(\mathbb{R}^n)$ with $N \geq L, \left[\frac{n}{2}\right] + 3$ on $\mathbb{R}^n \setminus \{0\}$ which satisfies

$$\begin{array}{lll} |\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\delta - |\gamma|}, & 0 < |\xi| \leq 1, & |\gamma| = L, \\ |\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\beta - |\gamma|}, & |\xi| > 1, & 2 \leq |\gamma| \leq \left[\frac{n}{2}\right] + 3. \end{array}$$

Suppose also that $1 \leq p, q \leq \infty$, $s_i \in \mathbb{R}$, $\alpha \in [0,1]$, for $i = 1, 2$ and satisfies $s_1 - s_2 \geq |\mathcal{S}_p(\beta)|$, then

$$\left|\left|e^{i\mu(D)} f\right|\right|_{M_{p,q}^{s_2,\alpha}(\mathbb{R}^n)} \leq C\, ||f||_{M_{p,q}^{s_1,\alpha}(\mathbb{R}^n)},$$

where the constant C is independent of f.

Zhao, Chen, and Guo [29] also developed necessary and sufficient conditions on a similar multiplier theorem.

2.4 Embedding Properties

There are some embedding relationships between these three function spaces, but such relationships are not clear cut. The various embedding properties between these three spaces are dependent on the essential parameters that define the function space. In this section we explore the nature of these embedding properties. The most obvious relationship is between the modulation space $M_{p,q}^s(\mathbb{R}^n)$ and the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ when $\alpha = 0$, which is stated as

$$M_{p,q}^s(\mathbb{R}^n) = M_{p,q}^{s,0}(\mathbb{R}^n).$$

This is an immediate result from the definition of the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.

The next two embedding properties are ones between the modulation space $M_{p,q}^s(\mathbb{R}^n)$, and the Besov space $B_{p,q}^s(\mathbb{R}^n)$. These results state under what conditions you need for $M_{p,q}^s(\mathbb{R}^n)$ to be embedded into $B_{p,q}^s(\mathbb{R}^n)$, and for the reverse direction. These results were explored by Wang and Hudzik in [27].

Proposition 2.4.1. *([27]) Define* $\sigma(p,q)$ *by*

$$\sigma(p,q) = \max\left\{0, n\left(\frac{1}{\min\{p,p'\}} - \frac{1}{q}\right)\right\}. \tag{2.10}$$

Let $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$*, then*

$$M_{p,q}^{s+\sigma(p,q)}(\mathbb{R}^n) \subset B_{p,q}^s(\mathbb{R}^n).$$

Similarly, Wang and Hudzik in [27] found conditions for the reverse direction which is summed up as the following:

Proposition 2.4.2. *([27]) Define* $\tau(p,q)$ *as*

$$\tau(p,q) = \begin{cases} \max\left\{0, n\left(\frac{1}{q} - \frac{1}{\max\{p,p'\}}\right)\right\}, & \text{when } 1 \leq p,q \leq \infty, \\ \frac{n}{q}, & \text{when } 0 < p < 1 \text{ or } 0 < q < 1. \end{cases}$$

For $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$*, then*

$$B_{p,q}^{s+\tau(p,q)}(\mathbb{R}^n) \subset M_{p,q}^s(\mathbb{R}^n).$$

The next embedding was obtained in Gröbner's Ph.D. thesis [15] and relates the Besov space $B_{p,q}^s(\mathbb{R}^n)$ to the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.

Proposition 2.4.3. *([15]) For* $1 \leq p, q \leq \infty$*,* $s \in \mathbb{R}$*, and* $0 \leq \alpha \leq 1$*, we have*

$$B_{p,q}^{s+\frac{(1-\alpha)n}{q}}(\mathbb{R}^n) \subseteq M_{p,q}^{s,\alpha}(\mathbb{R}^n).$$

Furthermore, for q' *that satisfies* $\frac{1}{q} + \frac{1}{q'} = 1$ *we have*

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) \subseteq B_{p,q}^{s-\frac{(1-\alpha)n}{q'}}(\mathbb{R}^n).$$

Note Proposition 2.4.3 shows that as $\alpha \to 1^-$ we obtain

$$B_{p,q}^s(\mathbb{R}^n) \subseteq M_{p,q}^{s,1}(\mathbb{R}^n) \subseteq B_{p,q}^s\mathbb{R}^n),$$

which gives us justification for saying

$$M_{p,q}^{s,1}(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n).$$

There are a few other embedding properties. Using Proposition 2.1.3 from Chapter 2 Section 2.1 we have the following.

Corollary 2.4.4. *The following embedding holds*

$$M_{p,q}^{s+\sigma(p,q)}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n),$$

where σ is defined in equation (2.10)*. Furthermore, we the following embedding holds*

$$M_{p,1}^{s+\sigma(p,q)}(\mathbb{R}^n) \subset W^{p,k}(\mathbb{R}^n).$$

Proof. These embeddings follow directly from Propositions 2.1.3 and 2.4.1. □

Han and Wang [18] showed another embedding property between the Besov space $B_{p,q}^s(\mathbb{R}^n)$ and the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$. Recall the definition of $R(p, q; \alpha_1, \alpha_2)$, equation (2.9) in Section 2.3.

Proposition 2.4.5. *([18]) Let $0 \leq \alpha < 1$. Then $B_{p,q}^{s_1}(\mathbb{R}^n) \subset M_{p,q}^{s_2,\alpha}$ holds if and only if $s_1 \geq s_2 + R(p, q; 1, \alpha)$. Conversely, $M_{p,q}^{s_1,\alpha}(\mathbb{R}^n) \subset B_{p,q}^{s_2}(\mathbb{R}^n)$ holds if and only if $s_1 \geq s_2 + R(p, q; \alpha, 1)$.*

With the above ideas we can see that the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ serves as more of an intermediate space. That means that as α varies from 0 to 1, the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$ begins to "look" like the modulation space $M_{p,q}^s(\mathbb{R}^n)$, or the Besov space $B_{p,q}^s(\mathbb{R}^n)$ respectively. Note that the term intermediate is not to be confused with the idea that there is a clear cut embedding between the three spaces that takes the form of $M_{p,q}^s(\mathbb{R}^n) \subset M_{p,q}^{s,\alpha}(\mathbb{R}^n) \subset B_{p,q}^s(\mathbb{R}^n)$. The above results makes this point very clear.

See Figure 2.4 for the classic picture that relates the three function spaces $M_{p,q}^s(\mathbb{R}^n)$, $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$, and $B_{p,q}^s(\mathbb{R}^n)$ when $p = q$ and an additional function space $H^s(\mathbb{R}^n)$, which is defined as

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : ||f||_{H^s(\mathbb{R}^n)} < \infty \right\},$$

where $||f||_{H^s(\mathbb{R}^n)}$ is defined as

$$||f||_{H^s(\mathbb{R}^n)} = \left|\left|(I - \Delta)^{\frac{s}{2}} f\right|\right|_{L^2(\mathbb{R}^n)}.$$

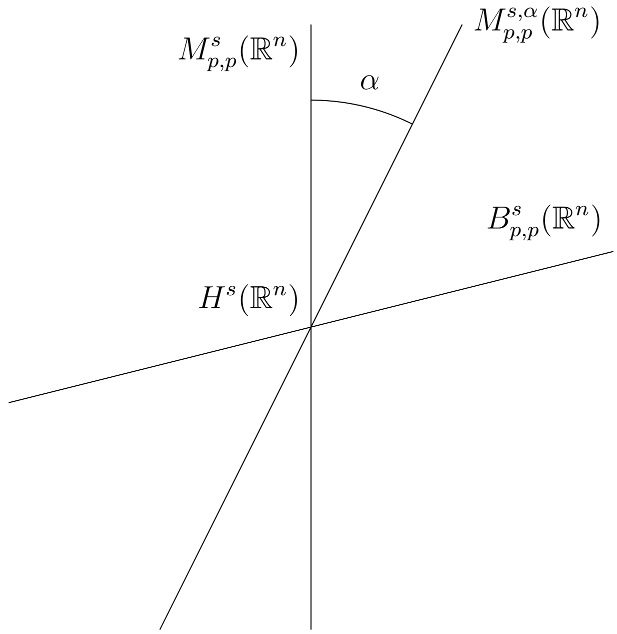

Figure 2.4: Relationship between $M^{s}_{p,p}(\mathbb{R}^n)$, $M^{s,\alpha}_{p,p}(\mathbb{R}^n)$, $B^{s}_{p,p}(\mathbb{R}^n)$, and $H^s(\mathbb{R}^n)$.

The diagram above illustrates how the parameter α acts a "tuner" that can produce a suitable decomposition of $\mathbb{R}^n$ that is in between the decomposition of the modulation space's $M^{s}_{p,q}(\mathbb{R}^n)$ uniformed rectangles, and the Besov space's $B^{s}_{p,q}(\mathbb{R}^n)$ annuli of size $[2^{j-1}, 2^j)$. The diagram also show that all four spaces coincide in the case of

$$M^{s}_{2,2}(\mathbb{R}^n) = M^{s,\alpha}_{2,2}(\mathbb{R}^n) = B^{s}_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n).$$

This further adds to our understanding of the behavior of these function spaces.

Chapter 3

Asymptotic Estimates

3.1 Statement of Main Results

Here we state the main results that quantify the asymptotic behavior for several Fourier Multipliers. For the remainder of the theorems we will suppose that $0 \leq \alpha < 1$. The following two theorems specifically deal with the Cauchy problem for dispersive equation.

Theorem 3.1.1. *(Trulen) If $\alpha_o = 1$, $1 \leq p, q, \leq \infty$, and $t > 1$, then*

$$\left|\left|e^{it|\Delta|^{\frac{\alpha_o}{2}}} f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \, ||f||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \, ||f||_{M^{s+\beta(\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma > 0$ and $\beta(\alpha)$ is

$$\beta(\alpha) = n\alpha\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{3.1}$$

Theorem 3.1.2. *(Trulen) If $\frac{1}{2} < \alpha_o$ with $\alpha_o \neq 1$, $1 \leq p, q \leq \infty$, and $t > 1$, then*

$$\left|\left|e^{it|\Delta|^{\frac{\alpha_o}{2}}} f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \, ||f||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \, ||f||_{M^{s+\beta(\alpha_o,\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma > 0$ and $\beta(\alpha_o, \alpha)$ is defined as

$$\beta(\alpha_o, \alpha) = n(\alpha_o - 2 + 2\alpha)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{3.2}$$

The first thing to note is that these estimates are in line with the results found by Chen, Fan, and Sun [5] when $\alpha = 0$. The second thing to note is that these estimates are similar to those found by Zhao, Chen, and Guo [29]. Since we used different methods to obtain these results, our estimates are a little more general than [29].

We also obtain similar results for the generalized half Klein-Gordon equation. The results are as follows:

Theorem 3.1.3. *(Trulen) For $\alpha_o \geq 1$, $1 \leq p, q \leq \infty$, and $t > 1$,*

$$\left|\left|e^{it(I-\Delta)^{\frac{\alpha_o}{2}}} f\right|\right|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M_{p,q}^{s+\beta(\alpha_o,\alpha),\alpha}(\mathbb{R}^n)},$$

where $\gamma > 0$ and $\beta(\alpha_o, \alpha)$ is defined as in equation (3.2).

Both Theorems 3.1.2 and 3.1.3 say that in the α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$, as $t \to \infty$ the multipliers $t^{-n\left|\frac{1}{p}-\frac{1}{2}\right|} e^{it|\Delta|^{\frac{\alpha_o}{2}}}$ and $t^{-n\left|\frac{1}{p}-\frac{1}{2}\right|} e^{it(I-|\Delta|)^{\frac{\alpha_o}{2}}}$ gain a regularity $n(2 - 2\alpha - \alpha_o)\left|\frac{1}{p} - \frac{1}{2}\right|$ when $\frac{1}{2} < \alpha_o \leq 2(1-\alpha)$ and $\alpha_o \neq 1$ for the Cauchy problem for dispersive equation multiplier and when $1 \leq \alpha_o \leq 2(1-\alpha)$ for the generalized half Klein-Gordon equation multiplier. Both multipliers loss a regularity $n(\alpha_o - 2 + 2\alpha)\left|\frac{1}{p} - \frac{1}{2}\right|$ when $\alpha_o > 2(1-\alpha)$.

We also obtain asymptotic estimates for the Fourier multiplier $\Theta(t) = \dfrac{\sin(t(-\Delta)^{\frac{1}{2}})}{(-\Delta)^{\frac{1}{2}}}$ and $\Theta_K(t) = \dfrac{\sin(t(-\Delta)^{\frac{1}{2}})}{(-\Delta)^{\frac{1}{2}}}$ which has symbols $\dfrac{\sin(t|\xi|)}{|\xi|}$ and $\dfrac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}$ respectively. These estimates will be used to obtain a unique solution for a nonlinear wave and Klein-Gordon equations. The results are as follows.

Theorem 3.1.4. *(Trulen) For $1 \leq p \leq \infty$, and $t \geq 1$,*

$$||\Theta(t)g||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M_{p,q}^{s+\beta_1(\alpha),\alpha}(\mathbb{R}^n)},$$

where $\gamma > 0$ and $\beta_1(\alpha)$ is defined as

$$\beta_1(\alpha) = (\alpha n - 2)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{3.3}$$

Theorem 3.1.5. *(Trulen) Let $1 \leq p, q \leq \infty$, and $t \geq 1$. If there exists $N > 0$ such that*

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{L^1(\mathbb{R}^n)},$$

when $|k| < N$ and

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k \rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{L^1(\mathbb{R}^n)},$$

when $|k| \geq N$ with $b_1 \geq b_2 \geq 0$, d is a real number, then the following estimate holds:

$$||\Theta_K(t)g||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M_{p,q}^{s+\beta_1(\alpha),\alpha}(\mathbb{R}^n)},$$

where γ is any positive number, and β_1 is defined by

$$\beta_1(\alpha) = (n\alpha - 2)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Before we are able to prove these results there are some known theorems, as well as a new result relating L^p-estimates of a unimodular Fourier multiplier to its $M_{p,q}^{s,\alpha}$-estimate. These topics will be covered in the next section.

3.2 Existing Theory and New Estimates

First we will start with a survey of well known theorems that will be of interest to us. These results range from general analysis to harmonic analysis and interpolation theory.

Proposition 3.2.1. *(Schwartz Inequality [22]) Suppose f and g are integrable functions on $\mathbb{X}$, then the following holds*

$$\int_{\mathbb{X}} f(x)g(x)dx \leq \int_{\mathbb{X}} |f(x)|^2 dx \int_{\mathbb{X}} |g(x)|^2 dx.$$

Proposition 3.2.2. *(Bernstein Multiplier Theorem ([3])) Let $L > \dfrac{n}{2}$ be an integer and $H_\mu(f)$ be the Fourier multiplier defined as in equation (1.7) in Chapter 1 Section 1.3. Then the operator norm of H_μ on the space $L^p(\mathbb{R}^n)$ satisfies*

$$||H_\mu||_{L^p \to L^p} \preceq ||\mu||_{L^2}^{1-\frac{n}{2L}} \left(\sum_{i=1}^{n} \left|\left|\partial_{x_i}^L \mu\right|\right|_{L^2}^{\frac{n}{2L}} \right),$$

for all $1 \leq p \leq \infty$.

In the case when $\alpha_o = 1$ and after a few substitutions, the Bernstein Multiplier Theorem will produce the t and $\langle k \rangle$ factors, but using this proposition will produce singularities.

Proposition 3.2.3. *(Sobolev Embedding Theorem ([23])) Recall the Riesz potential* $I_\beta(f) = (-\Delta)^{-\frac{\beta}{2}}(f)$ *where* $0 < \beta < n$*, see Chapter 1 Section 1.4. If we have*

$$\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n},$$

then the following estimate holds

$$||I_\beta(f)||_{L^q} \leq A\,||f||_{L^p}\,.$$

On the surface the Sobolev Embedding Theorem says something about the L^q-estimate of the Riesz potential when applied to an L^p-function. This operator was defined and briefly discussed in Chapter 1 Section 1.4. Our use of the Sobolev Embedding Theorem is to view it as a statement about the L^q-estimate of a singular integral. This will prove useful when getting bounds on such integrals that are created when using the Bernstein Multiplier Theorem.

Proposition 3.2.4. *(Van der Corput Lemma ([24, 14, 20])) Let* $E \subset \mathbb{R}^n$ *denote an open set and* $\psi \in C_c^\infty(E)$*. If* $\phi \in C^\infty(E)$ *and the rank of the matrix*

$$(D_{x_j} D_{x_i} \phi(x))_{i,j=1}^n,$$

is at least $\rho > 0$ *for all* $x \in supp(\psi)$*, then we have*

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x) dx \right| \preceq |\lambda|^{-\frac{\rho}{2}}\,||\psi||_{C^{2n}}\,.$$

The Van der Corput Lemma is used when $\alpha_o > \dfrac{1}{2}$ and $\alpha_o \neq 1$, and when integration-by-parts fails to work.

Proposition 3.2.5. *(Riesz-Thorin Interpolation Theorem ([3, 20])) Let $p_0 \neq p_1$ and $q_0 \neq q_1$. Suppose T is a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ to $L^{q_0}(\mathbb{R}^n)$ with norm M_0 and from $L^{p_1}(\mathbb{R}^n)$ to $L^{q_1}(\mathbb{R}^n)$ with norm M_1, then T is bounded from $L^{p_\theta}(\mathbb{R}^n)$ to $L^{q_\theta}(\mathbb{R}^n)$ with norm M_θ such that*

$$M_\theta \leq M_0^{1-\theta} M_1^{\theta},$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \text{ and } \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

where $0 \leq \theta \leq 1$.

With the Riesz-Thorin Interpolation Theorem we are able to state and prove a proposition about how the L^p-estimates of a unimodular Fourier multiplier relate to its $M_{p,q}^{s,\alpha}$-estimates. For the next proposition let e^{itM} be the unimodular Fourier multiplier with symbol $e^{it\nu(\xi)}$ where $\nu(\xi)$ is a real-valued function. Note that in our case we will be interested in $e^{itM} = e^{it|\Delta|^{\frac{\alpha_o}{2}}}$ and $e^{it(I-|\Delta|)^{\frac{\alpha_o}{2}}}$ are unimodular Fourier multipliers with symbols $e^{it|\xi|^{\alpha_o}}$ and $e^{it(1+|\xi|^2)^{\frac{\alpha_o}{2}}}$ respectively.

Proposition 3.2.6. *(Trulen) Let $t > 0$ and $\square_k^\alpha$ be defined by equation (2.7) in Chapter 2 Section 2.3. Suppose there exists an $N > 0$ such that*

$$\left|\left|\square_k^\alpha e^{itM} f\right|\right|_{L^1} \preceq t^{b_1} \left|\left|f\right|\right|_{L^1}, \tag{3.4}$$

if $|k| < N$ and

$$\left|\left|\square_k^\alpha e^{itM} f\right|\right|_{L^1} \preceq t^{b_2} \langle k \rangle^d \left|\left|f\right|\right|_{L^1}, \tag{3.5}$$

if $|k| \geq N$, where $b_1 \geq b_2 \geq 0$ and d is a real number, then

$$\left|\left|e^{itM} f\right|\right|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left|f\right|\right|_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left|f\right|\right|_{M_{p,q}^{s+\beta,\alpha}(\mathbb{R}^n)},$$

where $\gamma \geq 0$ and β is defined as

$$\beta = 2d(1-\alpha)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Proof. By repeated use of Plancherel's Theorem 1.2.1 we have

$$\begin{aligned}
\left|\left|\square_k^\alpha e^{itM} f\right|\right|_{L^2} &= \left|\left|\mathscr{F}^{-1}\eta_k^\alpha(\cdot)\mathscr{F}(e^{itM} f)\right|\right|_{L^2} \\
&= \left|\left|\eta_k^\alpha(\cdot)\mathscr{F}(e^{itM} f)\right|\right|_{L^2} \\
&\preceq \left|\left|\mathscr{F}(e^{itM} f)\right|\right|_{L^2} \\
&= \left|\left|e^{itM} f\right|\right|_{L^2} \\
&= \left|\left|\mathscr{F}^{-1} e^{it\nu(\xi)}\mathscr{F} f\right|\right|_{L^2} \\
&= \left|\left|e^{it\nu(\xi)}\mathscr{F} f\right|\right|_{L^2} \\
&= \left|\left|\mathscr{F} f\right|\right|_{L^2} \\
&= \left|\left|f\right|\right|_{L^2}.
\end{aligned}$$

Thus we have

$$\left|\left|\square_k^\alpha e^{itM} f\right|\right|_{L^2} \preceq \left|\left|f\right|\right|_{L^2}. \tag{3.6}$$

Using estimate (3.4) from the hypothesis and estimate (3.6) we have by the Riesz-Thorin Interpolation Theorem, Proposition 3.2.5

$$\left|\left|\square_k^\alpha e^{itM} f\right|\right|_{L^p} \preceq \left(t^{b_1}\left|\left|f\right|\right|_{L^1}\right)^\theta \left|\left|f\right|\right|_{L^2}^{1-\theta},$$

when $|k| < N$ and

$$\frac{1}{p} = \frac{1-\theta}{2} + \theta.$$

This implies that

$$\theta = 2\left(\frac{1}{p} - \frac{1}{2}\right).$$

Therefore, we have

$$\begin{aligned}
\left|\left|\square_k^\alpha e^{itM} f\right|\right|_{L^p} &\preceq \left(t^{b_1}\left|\left|f\right|\right|_{L^1}\right)^\theta \left|\left|f\right|\right|_{L^2}^{1-\theta} \\
&= \left(t^{b_1}\left|\left|f\right|\right|_{L^1}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \left|\left|f\right|\right|_{L^2}^{1-2\left(\frac{1}{p}-\frac{1}{2}\right)} \\
&= t^{2b_1\left(\frac{1}{p}-\frac{1}{2}\right)} \left|\left|f\right|\right|_{L^1}^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \left|\left|f\right|\right|_{L^2}^{1-2\left(\frac{1}{p}-\frac{1}{2}\right)} \\
&\preceq t^{2b_1\left(\frac{1}{p}-\frac{1}{2}\right)} \left|\left|f\right|\right|_{L^p},
\end{aligned}$$

since $f \in \mathscr{S}'$.

Using estimate (3.5) from the hypothesis and estimate (3.6) we have again by Riesz-Thorin Interpolation Theorem, Proposition 3.2.5

$$\begin{aligned}||\Box_k^\alpha e^{itM} f||_{L^p} &\preceq \left(t^{b_2}|k|^d ||f||_{L^1}\right)^\theta ||f||_{L^2}^{1-\theta} \\ &= \left(t^{b_2}|k|^d ||f||_{L^1}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} ||f||_{L^2}^{1-2\left(\frac{1}{p}-\frac{1}{2}\right)} \\ &= t^{2b_2\left(\frac{1}{p}-\frac{1}{2}\right)} \langle k\rangle^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} ||f||_{L^1}^{2\left(\frac{1}{p}-\frac{1}{2}\right)} ||f||_{L^2}^{1-2\left(\frac{1}{p}-\frac{1}{2}\right)} \\ &\preceq t^{2b_2\left(\frac{1}{p}-\frac{1}{2}\right)} \langle k\rangle^{2d\left(\frac{1}{p}-\frac{1}{2}\right)} ||f||_{L^p},\end{aligned}$$

if $|k| \geq N$. Using duality we get

$$||\Box_k^\alpha e^{itM} f||_{L^p} \preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{L^p},$$

and

$$||\Box_k^\alpha e^{itM} f||_{L^p} \preceq t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{2d\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{L^p},$$

for all $1 \leq p \leq \infty$.

Since we have

$$1 = \sum_{|k|=-\infty}^{\infty} \eta_k^\alpha(\xi),$$

then by almost orthogonality for the family $\{\Box_k^\alpha\}$ we have

$$\Box_k^\alpha e^{itM} = \sum_{|l|\leq \gamma_{C,k}} \Box_{k+l}^\alpha \Box_k^\alpha e^{itM}.$$

Now for $|k| < N$ we have

$$\begin{aligned}||\Box_k^\alpha e^{itM} f||_{L^p} &= \left|\left|\sum_{|l|\leq \gamma_{C,k}} \Box_{k+l}^\alpha \Box_k^\alpha e^{itM} f\right|\right|_{L^p} \\ &\leq \sum_{|l|\leq \gamma_{C,k}} ||\Box_{k+l}^\alpha e^{itM} \Box_k^\alpha f||_{L^p} \\ &\preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} ||\Box_k^\alpha f||_{L^p}.\end{aligned}$$

For $|k| \geq N$ we have

$$\begin{aligned} \left|\left|\Box_k^\alpha e^{itM} f\right|\right|_{L^p} &= \left|\left| \sum_{|l|\leq \gamma_{C,k}} \Box_{k+l}^\alpha \Box_k^\alpha e^{itM} f \right|\right|_{L^p} \\ &\leq \sum_{|l|\leq \gamma_{C,k}} \left|\left|\Box_{k+l}^\alpha e^{itM} \Box_k^\alpha f\right|\right|_{L^p} \\ &\preceq t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{2d\left|\frac{1}{p}-\frac{1}{2}\right|} ||\Box_k^\alpha f||_{L^p}, \end{aligned}$$

where $\gamma_{C,k}$ is a positive constant that depends on C and k.

Using the definition of α-modulation spaces we have

$$\begin{aligned} \left|\left|e^{itM} f\right|\right|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} &= \left|\left| \langle k\rangle^{\frac{s}{1-\alpha}} \left|\left|\Box_k^\alpha e^{itM} f\right|\right|_{L^p} \right|\right|_{\ell^q} \\ &= \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \left|\left|\Box_k^\alpha e^{itM} f\right|\right|_{L^p}^q \right)^{\frac{1}{q}} \\ &\leq \left(\sum_{|k|<N} \langle k\rangle^{\frac{sq}{1-\alpha}} \left|\left|\Box_k^\alpha e^{itM} f\right|\right|_{L^p}^q \right)^{\frac{1}{q}} \\ &\quad + \left(\sum_{|k|\geq N} \langle k\rangle^{\frac{sq}{1-\alpha}} \left|\left|\Box_k^\alpha e^{itM} f\right|\right|_{L^p}^q \right)^{\frac{1}{q}} \\ &\preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|<N} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\Box_k^\alpha f||_{L^p}^q \right)^{\frac{1}{q}} \\ &\quad + t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|\geq N} \langle k\rangle^{2dq\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\Box_k^\alpha f||_{L^p}^q \right)^{\frac{1}{q}} \\ &\preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|<N} ||\Box_k^\alpha f||_{L^p}^q \right)^{\frac{1}{q}} \\ &\quad + t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|\geq N} \langle k\rangle^{\frac{q}{1-\alpha}\left(2d(1-\alpha)\left|\frac{1}{p}-\frac{1}{2}\right|+s\right)} ||\Box_k^\alpha f||_{L^p}^q \right)^{\frac{1}{q}} \\ &\preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M_{p,q}^{s+\beta,\alpha}(\mathbb{R}^n)}, \end{aligned}$$

where $\gamma \geq 0$ and β is defined as

$$\beta = 2d(1-\alpha)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

This completes the proof. □

Note that this result corresponds with a result found by Chen, Fan, and Sun [5] when $\alpha = 0$, i.e., for the modulation space $M^s_{p,q}(\mathbb{R}^n)$.

The last two known results we need are ones that were found by Chen, Fan, and Sun [5].

Lemma 3.2.7. *([5]) For $|k| = 0$ we have for $\alpha_o > \frac{1}{2}$*

$$\left|\left|\mathscr{F}^{-1}(\eta^\alpha_k(\xi)e^{it|\xi|^{\alpha_o}})\right|\right|_{L^1} \preceq t^{\frac{n}{2}}.$$

Lemma 3.2.8. *([5]) When $|k| = 0$ we have the estimate for $1 \leq p \leq \infty$*

$$||\Box^\alpha_0 \Theta(t) g||_{L^p} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||\Box^\alpha_0 g||_{L^p}.$$

This result was originally stated in the case of the modulation space $M^s_{p,q}(\mathbb{R}^n)$ using a function that satisfy condition (2.5) in Chapter 2 Section 2.2. In the case when $|k| = 0$ their results become equivalent to these results.

3.3 Proof for the Asymptotic Estimate for the Cauchy Problem for Dispersive Equation

Throughout the following proofs let ξ^α_k, $|\xi|^\alpha_k$, and $|\xi|^{\alpha,\alpha_o}_k$ denote the following

$$\xi^\alpha_k = \langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi + k), \tag{3.7}$$

$$|\xi|^\alpha_k = \langle k\rangle^{\frac{\alpha}{1-\alpha}}|\xi + k|, \tag{3.8}$$

$$|\xi|^{\alpha,\alpha_o}_k = \langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi + k|^{\alpha_o}. \tag{3.9}$$

To prove Theorems 3.1.1 and 3.1.2 there is an additional lemma and proposition that needs to be proven. The first is a technical lemma that describes the boundedness of various derivatives of the symbol $e^{it|\xi|^{\alpha_o}}$ with a translation and scaling of $\langle k\rangle^{\frac{\alpha}{1-\alpha}}|\xi + k|$.

Lemma 3.3.1. *(Trulen) For all multi-indexes α' where $|\alpha'| = L \geq 1$, we have the following estimates for $D^{\alpha'} e^{it\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o}}$ for all $\alpha_o > 0$*

$$D^{\alpha'} e^{it\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o}} \preceq \begin{cases} t^L \langle k\rangle^{\frac{L\alpha_o\alpha}{1-\alpha}} |\xi+k|^{\alpha_o - L} e^{it\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o}}, & \text{if } |\xi+k| < 1, \\ t^L \langle k\rangle^{\frac{L\alpha_o\alpha}{1-\alpha}} |\xi+k|^{L\alpha_o - L} e^{it\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o}}, & \text{if } |\xi+k| \geq 1. \end{cases}$$

Proof. A simple calculation shows that for all multi-index α', $D^{\alpha'} e^{it|\xi|_k^{\alpha,\alpha_o}}$ is made up of terms in the form of

$$\sum_{j_1,j_2} C_{\alpha_o}(it\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}})^B |\xi+k|^{j_1\alpha_o - 2j_2} \prod_{h=1}^{n} (\xi_h + k_h)^{\beta_h} e^{it|\xi|_k^{\alpha,\alpha_o}},$$

plus the term

$$C_{\alpha_o}(it\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}})^L |\xi+k|^{L\alpha_o - 2L} \prod_{h=1}^{n} (\xi_h + k_h)^{\gamma_h} e^{it|\xi|_k^{\alpha,\alpha_o}},$$

where C_{α_o} is a constant dependent on α_o, $|\alpha'| = L$, $B \leq L$, $j_1 \leq j_2$, $1 \leq j_1 \leq L-1$, $\left\lceil \frac{L}{2} \right\rceil \leq j_2 \leq L$ with the following relationship

$$\beta_1 + \cdots + \beta_n - 2j_2 = -L, \text{ and } \gamma_1 + \cdots + \gamma_n - 2L = -L.$$

Thus it follows that

$$D^{\alpha'} e^{it|\xi|_k^{\alpha,\alpha_o}} \preceq t^L \langle k\rangle^{\frac{\alpha_o\alpha L}{1-\alpha}} \left(|\xi+k|^{L\alpha_o - L} + \sum_{j_1} |\xi+k|^{j_1\alpha_o - L} \right) e^{it|\xi|_k^{\alpha,\alpha_o}}.$$

Now if $|\xi+k| < 1$ we have

$$\begin{aligned} D^{\alpha'} e^{it|\xi|_k^{\alpha,\alpha_o}} &\preceq t^L \langle k\rangle^{\frac{\alpha_o\alpha L}{1-\alpha}} \left(|\xi+k|^{L\alpha_o - L} + \sum_{j_1} |\xi+k|^{j_1\alpha_o - L} \right) e^{it|\xi|_k^{\alpha,\alpha_o}} \\ &\preceq t^L \langle k\rangle^{\frac{\alpha_o\alpha L}{1-\alpha}} \left(|\xi+k|^{L\alpha_o - L} + |\xi+k|^{\alpha_o - L} \right) e^{it|\xi|_k^{\alpha,\alpha_o}} \\ &\preceq t^L \langle k\rangle^{\frac{\alpha_o\alpha L}{1-\alpha}} |\xi+k|^{\alpha_o - L} \left(|\xi+k|^{(L-1)\alpha_o + 1} \right) e^{it|\xi|_k^{\alpha,\alpha_o}} \\ &\preceq t^L \langle k\rangle^{\frac{\alpha_o\alpha L}{1-\alpha}} |\xi+k|^{\alpha_o - L} e^{it|\xi|_k^{\alpha,\alpha_o}}. \end{aligned}$$

Now if $|\xi + k| \geq 1$ then

$$\begin{aligned} D^{\alpha'} e^{it|\xi|_k^{\alpha,\alpha_o}} &\preceq t^L \langle k \rangle^{\frac{\alpha_o \alpha L}{1-\alpha}} \left(|\xi + k|^{L\alpha_o - L} + \sum_{j_1} |\xi + k|^{j_1 \alpha_o - L} \right) e^{it|\xi|_k^{\alpha,\alpha_o}} \\ &\preceq t^L \langle k \rangle^{\frac{\alpha_o \alpha L}{1-\alpha}} \left(|\xi + k|^{L\alpha_o - L} + |\xi + k|^{(L-1)\alpha_o - L} \right) e^{it|\xi|_k^{\alpha,\alpha_o}} \\ &\preceq t^L \langle k \rangle^{\frac{\alpha_o \alpha L}{1-\alpha}} |\xi + k|^{L\alpha_o - L} \left(1 + |\xi + k|^{-\alpha_o} \right) e^{it|\xi|_k^{\alpha,\alpha_o}} \\ &\preceq t^L \langle k \rangle^{\frac{\alpha_o \alpha L}{1-\alpha}} |\xi + k|^{L\alpha_o - L} e^{it|\xi|_k^{\alpha,\alpha_o}}. \end{aligned}$$

This completes the proof. □

With $\alpha_o = 1$ the statement and proof of the next proposition will give us Theorem 3.1.1.

Proposition 3.3.2. *(Trulen) Let η_k^α be a function defined that satisfy conditions* (2.6) *in Chapter 2 Section 2.3 and* $t \geq 1$. *If* $\alpha_o = 1$, *then we have*

$$\left|\left| \mathscr{F}^{-1}(\eta_k^\alpha(\xi) e^{it|\xi|^{\alpha_o}}) \right|\right|_{L^1} \preceq t^{\frac{n}{2}},$$

for $|k| = 0$ *and*

$$\left|\left| \mathscr{F}^{-1}(\eta_k^\alpha(\xi) e^{it|\xi|}) \right|\right|_{L^1} \preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\alpha}{2(1-\alpha)}},$$

for $|k| \neq 0$.

Proof. For $|k| = 0$, the result follows immediately from Lemma 3.2.7.

Now suppose $|k| \neq 0$ and let $L = \dfrac{n+1}{2}$ when n is odd and $L = \dfrac{n+2}{2}$ when n is even. First, making the substitutions of $\xi = \langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi' + k)$ followed by the substitution $x = \dfrac{x'}{\langle k\rangle^{\frac{\alpha}{1-\alpha}}}$, then applying Lemmas 3.2.2 and 3.3.1 we get

$$\begin{aligned}
\left|\left|\mathscr{F}^{-1}\left(\eta_k^\alpha e^{it|\xi|}\right)\right|\right|_{L^1} &= \int_{\mathbb{R}^n} \left|\mathscr{F}^{-1}\left(\eta_k^\alpha(\xi)e^{it|\xi|}\right)(x)\right| dx \\
&= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \eta_k^\alpha(\xi)e^{it|\xi|}e^{ix\xi}d\xi\right| dx \\
&= \langle k\rangle^{\frac{\alpha}{1-\alpha}} \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \eta_k^\alpha\left(\xi_k^\alpha\right) e^{i\left(t|\xi|_k^\alpha + x\xi_k^\alpha\right)}d\xi\right| dx \\
&= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \eta_k^\alpha\left(\xi_k^\alpha\right) e^{i\left(t|\xi|_k^\alpha + x(\xi+k)\right)}d\xi\right| dx \\
&= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \eta_k^\alpha\left(\xi_k^\alpha\right) e^{i\left(t|\xi|_k^\alpha + x\xi\right)}d\xi\right| dx \\
&= \left|\left|\mathscr{F}^{-1}\left(\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^\alpha}\right)\right|\right|_{L^1} \\
&\preceq \left|\left|\eta_k^\alpha\left(\xi_k^\alpha\right)\right|\right|_{L^2}^{1-\frac{n}{2L}} \sum_{|\delta|\le L}\left|\left|D^\delta\left(\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^\alpha}\right)\right|\right|_{L^2}^{\frac{n}{2L}} \\
&\preceq \sum_{|\delta|\le L}\left|\left|\sum_{\beta\le\delta} D^{\delta-\beta}\eta_k^\alpha\left(\xi_k^\alpha\right)t^{|\beta|}\langle k\rangle^{\frac{|\beta|\alpha}{1-\alpha}}|\xi+k|^{1-|\beta|}e^{it|\xi|_k^\alpha}\right|\right|_{L^2(\{\xi:|\xi+k|<1\})}^{\frac{n}{2L}} \\
&\quad + \sum_{|\delta|\le L}\left|\left|\sum_{\beta\le\delta} D^{\delta-\beta}\eta_k^\alpha\left(\xi_k^\alpha\right)t^{|\beta|}\langle k\rangle^{\frac{|\beta|\alpha}{1-\alpha}}e^{it|\xi|_k^\alpha}\right|\right|_{L^2(\{\xi:|\xi+k|\ge1\})}^{\frac{n}{2L}}.
\end{aligned}$$

Note that

$$\left|\left|\eta_k^\alpha\left(\xi_k^\alpha\right)\right|\right|_{L^2}^{1-\frac{n}{2L}} \preceq 1,$$

since $\eta_k^\alpha(\xi_k^\alpha)$ has support $|\xi| < C$. Now for the first norm note

$$1 - \frac{n+1}{2} = \frac{1-n}{2} > -n,$$

when n is odd and

$$1 - \frac{n+2}{2} = -\frac{n}{2} > -n,$$

when n is even. Then by Sobolev Embedding Theorem, Proposition 3.2.3, it follows

$$\begin{aligned}
\sum_{|\delta|\le L}\left|\left|\sum_{\beta\le\delta} D^{\delta-\beta}\eta_k^\alpha\left(\xi_k^\alpha\right)t^{|\beta|}\left\langle k\right\rangle^{\frac{|\beta|\alpha}{1-\alpha}}|\xi+k|^{1-|\beta|}e^{it\xi_k^\alpha}\right|\right|^{\frac{n}{2L}}_{L^2(\{\xi:|\xi+k|<1\})} \\
\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n\alpha}{2(1-\alpha)}}\left|\left|\eta_k^\alpha\left(\xi_k^\alpha\right)|\xi+k|^{1-L}e^{it|\xi|_k^\alpha}\right|\right|^{\frac{n}{2L}}_{L^2(\{\xi:|\xi+k|<1\})} \\
\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n\alpha}{2(1-\alpha)}}\left|\left|\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^\alpha}\right|\right|^{\frac{n}{2L}}_{L^p(\{\xi:|\xi+k|<1\})} \\
\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n\alpha}{2(1-\alpha)}},
\end{aligned}$$

where p satisfies

$$\frac{1}{2}=\frac{1}{p}-\frac{1-L}{n}.$$

For the second norm it is clear

$$\begin{aligned}
\sum_{|\delta|\le L}\left|\left|\sum_{\beta\le\delta} D^{\delta-\beta}\eta_k^\alpha\left(\xi_k^\alpha\right)t^{|\beta|}\left\langle k\right\rangle^{\frac{|\beta|\alpha}{1-\alpha}}e^{i|\xi|_k^\alpha}\right|\right|^{\frac{n}{2L}}_{L^2(\{\xi:|\xi+k|\ge1\})} \\
\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n\alpha}{2(1-\alpha)}}\left|\left|\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^\alpha}\right|\right|^{\frac{n}{2L}}_{L^2(\{\xi:|\xi+k|\ge1\})} \\
\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n\alpha}{2(1-\alpha)}}.
\end{aligned}$$

This completes the proof. □

To obtain Theorem 3.1.2 we need to prove the next proposition.

Proposition 3.3.3. *(Trulen) If $\alpha_o>\frac{1}{2}$ with $\alpha_o\neq1$ and $t\ge1$, then*

$$\left|\left|\mathscr{F}^{-1}(\eta_k^\alpha(\xi)e^{it|\xi|^{\alpha_o}})\right|\right|_{L^1}\preceq t^{\frac{n}{2}},$$

for $|k|=0$ and

$$\left|\left|\mathscr{F}^{-1}(\eta_k^\alpha(\xi)e^{it|\xi|^{\alpha_o}})\right|\right|_{L^1}\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n(\alpha_o-2+2\alpha)}{2(1-\alpha)}},$$

for $|k|\neq0$

Proof. If $\alpha_o>\frac{1}{2}$ with $\alpha_o\neq1$ and $|k|=0$, then by Lemma 3.2.7 we have

$$\left|\left|\mathscr{F}^{-1}(\eta_k^\alpha(\xi)e^{it|\xi|^{\alpha_o}})\right|\right|_{L^1}\preceq t^{\frac{n}{2}}.$$

Suppose that $|k| \neq 0$. Then, making the substitution of $\xi = \langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi' + k)$ followed by the substitution $x = \dfrac{x'}{\langle k \rangle^{\frac{\alpha}{1-\alpha}}}$ we have

$$\begin{aligned}
\left|\left|\mathscr{F}^{-1}\left(\eta_k^\alpha e^{it|\xi|^{\alpha_o}}\right)\right|\right|_{L^1} &= \int_{\mathbb{R}^n} \left|\mathscr{F}^{-1}\left(\eta_k^\alpha(\xi) e^{it|\xi|^{\alpha_o}}\right)(x)\right| dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_k^\alpha(\xi) e^{it|\xi|^{\alpha_o}} e^{ix\xi} d\xi \right| dx \\
&= \langle k \rangle^{\frac{\alpha}{1-\alpha}} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_k^\alpha(\xi_k^\alpha) e^{i\left(t|\xi|_k^{\alpha,\alpha_o} + x\xi_k^\alpha\right)} d\xi \right| dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_k^\alpha(\xi_k^\alpha) e^{i\left(t|\xi|_k^{\alpha,\alpha_o} + x(\xi+k)\right)} d\xi \right| dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta_k^\alpha(\xi_k^\alpha) e^{i\left(t|\xi|_k^{\alpha,\alpha_o} + x\xi\right)} d\xi \right| dx.
\end{aligned}$$

Define Φ as

$$\Phi = t|\xi|_k^{\alpha,\alpha_o} + x\xi.$$

Then the first two derivatives are

$$\frac{\partial \Phi}{\partial \xi_i} = \alpha_o t \langle k \rangle^{\frac{\alpha\alpha_o}{1-\alpha}} |\xi + k|^{\alpha_o - 2}(\xi_i + k_i) + x_i,$$

and $\dfrac{\partial^2 \Phi}{\partial \xi_i \partial \xi_j}$ is equal to

$$\alpha_o t \langle k \rangle^{\frac{\alpha\alpha_o}{1-\alpha}} |\xi + k|^{\alpha_o - 2} + \alpha_o(\alpha_o - 2) t \langle k \rangle^{\frac{\alpha\alpha_o}{1-\alpha}} |\xi + k|^{\alpha_o - 4}(\xi_i + k_i)^2,$$

if $j = i$ and

$$\alpha_o(\alpha_o - 2) t \langle k \rangle^{\frac{\alpha\alpha_o}{1-\alpha}} |\xi + k|^{\alpha_o - 4}(\xi_i + k_i)(\xi_j + k_j),$$

if $j \neq i$. Note that $\dfrac{\partial \Phi}{\partial \xi_i} = 0$ when

$$x_i = -\alpha_o t \langle k \rangle^{\frac{\alpha\alpha_o}{1-\alpha}} |\xi + k|^{\alpha_o - 2}(\xi_i + k_i),$$

or equivalently,

$$x = -\alpha_o t \langle k \rangle^{\frac{\alpha\alpha_o}{1-\alpha}} |\xi + k|^{\alpha_o - 2}(\xi + k).$$

For the case of $n = 2$ we have

$$\begin{aligned}
&\left|\det\left(D_{\xi_i}D_{\xi_j}\Phi\right)_{i,j=1}^2\right| \\
&= \Big|\Big(\alpha_o(\alpha_o-2)t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-4}(\xi_1+k_1)^2+\alpha_o t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-2}\Big)\times \\
&\quad \Big(\alpha_o(\alpha_o-2)t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-4|}(\xi_2+k_2)^2+\alpha_o t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-2}\Big) \\
&\quad -\alpha_o^2(\alpha_o-2)^2t^2\langle k\rangle^{\frac{2\alpha_o\alpha}{1-\alpha}}|\xi+k|^{2\alpha_o-8}(\xi_1+k_1)^2(\xi_2+k_2)^2\Big| \\
&= \Big|\alpha_o^2t^2\langle k\rangle^{\frac{2\alpha_o\alpha}{1-\alpha}}|\xi+k|^{2\alpha_o-4}\times \\
&\quad \big((\alpha_o-2)|\xi+k|^{-2}(\xi_1+k_1)^2+(\alpha_o-2)|\xi+k|^{-2}(\xi_2+k_2)^2+1\big)\Big| \\
&= \Big|\alpha_o^2t^2\langle k\rangle^{\frac{2\alpha_o\alpha}{1-\alpha}}|\xi+k|^{2\alpha_o-4}\times \\
&\quad \big((\alpha_o-2)|\xi+k|^{-2}\left((\xi_1+k_1)^2+(\xi_2+k_2)^2\right)+1\big)\Big| \\
&= \Big|\alpha_o^2t^2\langle k\rangle^{\frac{2\alpha_o\alpha}{1-\alpha}}|\xi+k|^{2\alpha_o-4}\left((\alpha_o-2)|\xi+k|^{-2}|\xi+k|^2+1\right)\Big| \\
&= \Big|\alpha_o^2t^2\langle k\rangle^{\frac{2\alpha_o\alpha}{1-\alpha}}|\xi+k|^{2\alpha_o-4}\left((\alpha_o-2)+1\right)\Big| \\
&= \Big|\alpha_o^2(\alpha_o-1)t^2\langle k\rangle^{\frac{2\alpha_o\alpha}{1-\alpha}}|\xi+k|^{2\alpha_o-4}\Big| \\
&= \left(t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-2}\right)^2\alpha_o^2|\alpha_o-1|.
\end{aligned}$$

Since $\alpha_o \neq 1$, we have

$$\left|\det\left(D_{\xi_i}D_{\xi_j}\Phi\right)_{i,j=1}^2\right| \geq t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2}.$$

Note, this calculation can be extended for $n \geq 3$.

Now define $C_i(k)$ and $D_i(k)$ as

$$C_i(k) = \alpha_o t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(|k_i|+C)\left(\sum_{j=1}^n(|k_j|+C)^2\right)^{\frac{\alpha_o-2}{2}},$$

and

$$D_i(k) = \alpha_o t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(|k_i|-C)\left(\sum_{j=1}^n(|k_j|-C)^2\right)^{\frac{\alpha_o-2}{2}}.$$

Now define the intervals F_i as the set of all $x_i \in \mathbb{R}^n$ such that

$$D_i(k) - t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2} < |x_i| < C_i(k) + t\langle k\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2},$$

$G_{i,j}$ as the set of all $x_i \in \mathbb{R}^n$ such that

$$C_i(k) + t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2} + j - 1 < |x_i| \le C_i(k) + t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2} + j,$$

and $H_{i,j}$ as the set of all $x_i \in \mathbb{R}^n$ such that

$$D_i(k) - t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2} - j < |x_i| \le D_i(k) + t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2} - j + 1.$$

Since

$$|x| = \alpha_o t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-2}|\xi+k| = \alpha_o t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-1},$$

it follows that

$$x_i \in F_i.$$

It also it follows that

$$\text{length}(F_i) \preceq t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2},$$

and

$$\text{length}(G_{i,j}) = \text{length}(H_{i,j}) = 1.$$

Now define $K_{i,j}$ as

$$K_{i,j} = G_{i,j} \cup H_{i,j}.$$

Then

$$\chi_{F_i}(x_i) + \sum_{j=1}^{\infty}\chi_{K_{i,j}}(x_i) = 1.$$

Thus we have

$$\begin{aligned}
&\left|\left|\mathscr{F}^{-1}(\eta_k^\alpha(\xi)e^{it|\xi|^{\alpha_o}})\right|\right|_{L^1} \preceq \\
&\int_{\mathbb{R}^n}\prod_{i=1}^{n}\chi_{F_i}(x_i)\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^{\alpha,\alpha_o}+x\xi}d\xi\right|dx+ \\
&\quad\sum_{j_*=1}^{n}\sum_{I_l}\int_{\mathbb{R}^n}\mathscr{A}_{I_l,j_*}(x)\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^{\alpha,\alpha_o}+x\xi}d\xi\right|dx+ \\
&\quad\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\chi_{K_{i,j_i}}(x_i)\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it|\xi|_k^{\alpha,\alpha_o}+x\xi}d\xi\right|dx \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $\mathscr{A}_{I_l,j_*}$ is the product of characteristic functions $\chi_{F_i}(x_i)$ and $\chi_{K_{i,j_*}}(x_i)$, where there is at least one $\chi_{F_i}(x_i)$ and at least one $\chi_{K_{i,j_*}}(x_i)$.

For I_1, with $\xi \in \operatorname{supp}\, \eta_k^\alpha\left(\xi_k^\alpha\right)$ and by Proposition 3.2.4 we have

$$\begin{aligned}
I_1 &\preceq \left(t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2}\right)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\chi_{F_i}(x_i)dx \\
&\preceq \left(t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2}\right)^{-\frac{n}{2}}\left(t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2}\right)^{n} \\
&= t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n\alpha_o\alpha}{2(1-\alpha)}}|k|^{\frac{n(\alpha_o-2)}{2}} \\
&\preceq t^{\frac{n}{2}}\left\langle k\right\rangle^{\frac{n(\alpha_o-2+2\alpha)}{2(1-\alpha)}}.
\end{aligned}$$

The last line follows since

$$\begin{aligned}
\left\langle k\right\rangle^{\frac{n\alpha_o\alpha}{2(1-\alpha)}}|k|^{\frac{n(\alpha_o-2)}{2}} &\preceq \left\langle k\right\rangle^{\frac{n\alpha_o\alpha}{2(1-\alpha)}+\frac{n(\alpha_o-2)}{2}} \\
&= \left\langle k\right\rangle^{\frac{n}{2}\frac{\alpha_o\alpha+(\alpha_o-2)(1-\alpha)}{1-\alpha}} \\
&= \left\langle k\right\rangle^{\frac{n}{2}\frac{\alpha_o-2+2\alpha}{1-\alpha}}.
\end{aligned}$$

Now note for $x \in K_{i,j}$ and $\xi \in \operatorname{supp}\, \eta_k^\alpha\left(\xi_k^\alpha\right)$ we have

$$\begin{aligned}
&\frac{\partial}{\partial\xi_l}\left(\frac{\eta_k^\alpha\left(\xi_k^\alpha\right)}{\frac{\partial}{\partial\xi_i}\Phi}\right) \\
&= \frac{\frac{\partial\Phi}{\partial\xi_i}\frac{\partial}{\partial\xi_l}\eta_k^\alpha\left(\xi_k^\alpha\right)-\eta_k^\alpha\left(\xi_k^\alpha\right)\frac{\partial^2\Phi}{\partial\xi_l\partial\xi_i}}{\left(\frac{\partial\Phi}{\partial\xi_i}\right)^2} \\
&\preceq \frac{\alpha_o t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-2}(\xi_i+k_i)+x_i-\frac{\partial^2\Phi}{\partial\xi_l\partial\xi_i}}{\left(\alpha_o t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|\xi+k|^{\alpha_o-2}(\xi_i+k_i)+x_i\right)^2} \\
&= O\left(\frac{1}{j+\sqrt{t}\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{2(1-\alpha)}}|k|^{\frac{\alpha_o-2}{2}}}+\frac{t\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{1-\alpha}}|k|^{\alpha_o-2}}{\left(j+\sqrt{t}\left\langle k\right\rangle^{\frac{\alpha_o\alpha}{2(1-\alpha)}}|k|^{\frac{\alpha_o-2}{2}}\right)^2}\right).
\end{aligned}$$

Thus, using integration-by-parts twice on each variable $\xi_1, \cdots, \xi_n$ we have

$$\begin{aligned} I_3 &\preceq \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{t^n \langle k \rangle^{\frac{n\alpha_o\alpha}{1-\alpha}} |k|^{n\alpha_o - 2n}}{\prod_{i=1}^{n} \left(j_i + \sqrt{t} \langle k \rangle^{\frac{\alpha_o\alpha}{2(1-\alpha)}} |k|^{\frac{\alpha_o - 2}{2}} \right)^2} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \chi_{K_{i,j_i}}(x_i) dx \\ &\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\alpha_o\alpha}{2(1-\alpha)}} |k|^{\frac{n(\alpha_o-2)}{2}} \\ &\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\alpha_o - 2 + 2\alpha)}{2(1-\alpha)}}. \end{aligned}$$

When $\xi \in \text{supp } \eta_k^\alpha(\xi_k^\alpha)$, I_2 is the sum of integrals of the form

$$\sum_{j_{l+1}=1}^{n} \cdots \sum_{j_n=1}^{n} \int_{\mathbb{R}^n} \mathscr{B}_l(x_{i_0}) \left| \int_{\mathbb{R}^n} \eta_k^\alpha(\xi_k^\alpha) e^{it|\xi|_k^{\alpha,\alpha_o} + x\xi} d\xi \right| dx,$$

where $\mathscr{B}_l(x_{i_0})$ is defined as

$$\mathscr{B}_l(x_{i_0}) = \prod_{i_0=1}^{l} \chi_{F_{i_o}}(x_{i_0}) \prod_{i_0=l+1}^{n} \chi_{K_{i_0,j_{i_0}}}(x_{i_0}).$$

By doing integration-by-parts twice on the variables $\xi_{l+1}, \cdots, \xi_n$, the above integral is bounded by

$$\begin{aligned} I_2 &\preceq \sum_{j_{l+1}=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{t^{n-l} \langle k \rangle^{\frac{(n-l)\alpha_o\alpha}{1-\alpha}} |k|^{(n-l)\alpha_o - 2n}}{\prod_{i=l+1}^{n} \left(j_i + \sqrt{t} \langle k \rangle^{\frac{\alpha_o\alpha}{2(1-\alpha)}} |k|^{\frac{\alpha_o - 2}{2}} \right)^2} \int_{\mathbb{R}^n} \mathscr{B}_l(x_{i_0}) dx \\ &\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\alpha_o\alpha}{2(1-\alpha)}} |k|^{\frac{n(\alpha_o-2)}{2}} \\ &\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\alpha_o - 2 + 2\alpha)}{2(1-\alpha)}}. \end{aligned}$$

This completes the proof. □

3.4 Proof for the Asymptotic Estimate for the Generalized Half Klein-Gordon Equation

Throughout these proofs let $(\xi)^{\alpha_o}$ and $(\xi)_k^{\alpha,\alpha_o}$ be defined as

$$(\xi)^{\alpha_o} = (1 + |\xi|^2)^{\frac{\alpha_o}{2}}, \text{ and} \tag{3.10}$$

$$(\xi)_k^{\alpha,\alpha_o} = (1 + \langle k \rangle^{\frac{2\alpha}{1-\alpha}} |\xi + k|^2)^{\frac{\alpha_o}{2}}. \tag{3.11}$$

Lemma 3.4.1. *(Trulen) For all multi-indexes α' where $|\alpha'| = L \geq 1$ we have the following estimates for $D^{\alpha'} e^{it(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\alpha_o}{2}}}$ for all $\alpha_o \geq 0$*

$$D^{\alpha'} e^{it(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\alpha_o}{2}}} \preceq t^L \langle k\rangle^{\frac{L\alpha_o\alpha}{1-\alpha}} |\xi+k|^{L\alpha_o - L} e^{it\left(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2\right)^{\frac{\alpha_o}{2}}},$$

for all $\xi \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$.

Proof. The α' derivative takes the form of

$$C_{\alpha_o} t^L \langle k\rangle^{\frac{2L\alpha}{1-\alpha}} (1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{L\alpha_o-2L}{2}} \prod_{h=1}^{n} (\xi_h+k_h)^{\gamma_h} e^{it(\xi)_k^{\alpha,\alpha_o}},$$

plus terms of the form

$$C_{\alpha_o} t^{\beta} \langle k\rangle^{\frac{2\beta\alpha}{1-\alpha}} (1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{j_1\alpha_o-2j_2}{2}} \prod_{h=1}^{n} (\xi_h+k_h)^{\beta_h} e^{it(\xi)_k^{\alpha,\alpha_o}},$$

where $j_1 \leq j_2$, $\beta \leq L$, $1 \leq j_1 \leq L-1$, $\left\lceil \frac{L}{2} \right\rceil \leq j_2 \leq L$, $\gamma_1 + \cdots + \gamma_n - 2L = -L$, and $\beta_1 + \cdots + \beta_n - 2L = -L$.

Since $1 + \langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2 \geq 1$ and

$$\frac{(j_1-L)\alpha_o}{2} < 0,$$

then it follows that

$$\begin{aligned}
& C_{\alpha_o} t^L \langle k\rangle^{\frac{2L\alpha}{1-\alpha}} (1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{L\alpha_o-2L}{2}} \prod_{h=1}^{n} (\xi_h+k_h)^{\gamma_h} e^{it(\xi)_k^{\alpha,\alpha_o}} \\
&\quad + \sum_{j_1,j_2} C_{\alpha_o} t^{\beta} \langle k\rangle^{\frac{2\beta\alpha}{1-\alpha}} (1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{j_1\alpha_o-2j_2}{2}} \prod_{h=1}^{n} (\xi_h+k_h)^{\beta_h} e^{it(\xi)_k^{\alpha,\alpha_o}} \\
&\preceq \langle k\rangle^{\frac{2L\alpha}{1-\alpha}} \left(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2\right)^{\frac{L\alpha_o-2L}{2}} \\
&\quad \times \left(\prod_{h=1}^{n} (\xi_h+k_h)^{\gamma_h} + \left(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2\right)^{\frac{(j_1-L)\alpha_o}{2}} \prod_{h=1}^{n} (\xi_h+k_h)^{\beta_h} \right) \\
&\preceq \langle k\rangle^{\frac{2L\alpha}{1-\alpha}} \left(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2\right)^{\frac{L\alpha_o-2L}{2}} |\xi+k|^L \\
&\sim \langle k\rangle^{\frac{2L\alpha}{1-\alpha}} \left(\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2\right)^{\frac{L\alpha_o-2L}{2}} |\xi+k|^L \\
&\preceq \langle k\rangle^{\frac{L\alpha_o\alpha}{1-\alpha}} |\xi+k|^{L\alpha_o-L}.
\end{aligned}$$

This completes the proof. $\square$

Proposition 3.4.2. *(Trulen) For $\alpha_o > \dfrac{1}{2}$ and $|k| = 0$, we have the following estimate:*

$$\left|\left|e^{it(I-\Delta)^{\frac{\alpha_o}{2}}}\eta_0^{\alpha}(\xi)\right|\right|_{L^1} \preceq t^{\frac{n}{2}}.$$

Proof. Let $L = \dfrac{n+1}{2}$ if n is odd and $L = \dfrac{n+2}{2}$ if n is even. First note that

$$\begin{aligned}\left|\left|e^{it(I-\Delta)^{\frac{\alpha_o}{2}}}\eta_0^{\alpha}(\xi)\right|\right|_{L^1} \preceq & \int_{|x|\le t}\left|\int_{\mathbb{R}^n}\eta_0^{\alpha}(\xi)e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx \\ & + \int_{|x|>t}\left|\int_{\mathbb{R}^n}\eta_0^{\alpha}(\xi)e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx.\end{aligned}$$

For the first norm by Schwartz's inequality, Proposition 3.2.1 and Plancherel's Theorem, Proposition 1.2.1, we have

$$\begin{aligned}&\int_{|x|\le t}\left|\int_{\mathbb{R}^n}\eta_0^{\alpha}(\xi)e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx \\ &\preceq \left(\int_{|x|\le t}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\left(\eta_0^{k}(\xi)e^{it(\xi)^{\alpha_o}}e^{ix\xi}\right)^2 d\xi\right)^{\frac{1}{2}} \\ &\preceq t^{\frac{n}{2}}\left|\left|\eta_0^{k}(\xi)\right|\right|_{L^2} \\ &\preceq t^{\frac{n}{2}}.\end{aligned}$$

For the second norm define E_t by

$$E_t = \{x \in \mathbb{R}^n : |x| > t\}.$$

For $i, j \in \{1, 2, \cdots, n\}$ define $E_{t,i}$ by

$$E_{t,i} = \{x \in E_t : |x_i| \ge |x_j| \text{ for all } j \ne i\}.$$

Now by integration-by-parts and Lemma 3.4.1 we have

$$\begin{aligned}
&\int_{|x|>t}\left|\int_{\mathbb{R}^n}\eta_0^\alpha(\xi)e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx\\
&\preceq\sum_{i=1}^n\int_{E_{t,i}}\left|\int_{\mathbb{R}^n}\eta_0^\alpha(\xi)e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx\\
&\preceq\sum_{i=1}^n\int_{E_{t,i}}\frac{1}{|x|^L}\left|\int_{\mathbb{R}^n}\partial_{\xi_i}^L\left(\eta_0^\alpha(\xi)e^{it(\xi)^{\alpha_o}}\right)e^{ix\xi}d\xi\right|dx\\
&\preceq t^L\int_{\mathbb{R}^n}\frac{1}{|x|^L}\left|\int_{\mathbb{R}^n}\sum_{\delta=1}^L\partial_{\xi_i}^{L-\delta}\eta_0^\alpha(\xi)|\xi|^{\beta\alpha_o-\beta}e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx\\
&\preceq t^L\int_{\mathbb{R}^n}\frac{1}{|x|^L}\left|\int_{\mathbb{R}^n}\eta_0^\alpha(\xi)|\xi|^{L\alpha_o-L}e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx.
\end{aligned}$$

In either case of n being odd or even, it follows that

$$2L(1-\alpha_o)=(n+1)(1-\alpha_o)<\frac{n+1}{2}<\frac{n}{2}+1.$$

Thus by Schwartz's inequality, Proposition 3.2.1, and noting that η_0^α has compact support $|\xi|<C$ and $|\xi|>1$ it follows that

$$\begin{aligned}
&t^L\int_{\mathbb{R}^n}\frac{1}{|x|^L}\left|\int_{\mathbb{R}^n}\eta_0^\alpha(\xi)|\xi|^{L\alpha_o-L}e^{it(\xi)^{\alpha_o}}e^{ix\xi}d\xi\right|dx\\
&\preceq t^L\left(\int_{|x|>t}\frac{dx}{|x|^{2L}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\eta_0^\alpha(\xi)|^2|\xi|^{2L(\alpha_o-1)}d\xi\right)^{\frac{1}{2}}\\
&\preceq t^{\frac{n}{2}}.
\end{aligned}$$

This completes the proof. □

Proposition 3.4.3. *(Trulen) For $|k|\neq 0$ and $t>1$. If $\alpha_o\geq 1$, then we have the following estimate:*

$$\left|\left|e^{it(I-\Delta)^{\frac{\alpha_o}{2}}}\eta_k^\alpha(\xi)\right|\right|_{L^1}\preceq t^{\frac{n}{2}}\langle k\rangle^{\frac{n(\alpha_o-2+2\alpha)}{2(1-\alpha)}}.$$

Proof. Suppose $k\neq 0$. Like Proposition 3.3.3, first make the substitution of $\xi=$

$\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi' + k)$ followed by the substitution $x = \dfrac{x'}{\langle k \rangle^{\frac{\alpha}{1-\alpha)}}}$ to get

$$\begin{aligned}\left\|\Box_k^\alpha e^{it(I-\Delta)^{\frac{\alpha_o}{2}}}\right\|_{L^1} &= \left\|\mathscr{F}^{-1}\left(\eta_k^\alpha(\xi)e^{it(1+|\xi|^2)^{\frac{\alpha_o}{2}}}\right)\right\|_{L^1} \\ &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\xi_k^\alpha\right)e^{it(\xi)_k^{\alpha,\alpha_o}}e^{ix(\xi+k)}d\xi\right|dx \\ &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\xi_k^\alpha\right)e^{i\left(t(\xi)_k^{\alpha,\alpha_o}+x\xi\right)}d\xi\right|dx.\end{aligned}$$

Define Φ as

$$\begin{aligned}\Phi &= t(\xi)_k^{\alpha,\alpha_o} + x\xi \\ &= \langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}} t\left(\langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o}{2}} + x\xi.\end{aligned}$$

Now we have

$$\frac{\partial\Phi}{\partial\xi_i} = \alpha_o t\langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(\xi_i+k_i)\left(\langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o-2}{2}} + x_i,$$

and we have $\dfrac{\partial^2\Phi}{\partial\xi_i\partial\xi_j}$ equal to

$$\alpha_o(\alpha_o-2)t\langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(\xi_i+k_i)(\xi_j+k_j)\left(\langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o-4}{2}},$$

if $i \neq j$ and

$$\begin{aligned}&\alpha_o(\alpha_o-2)t\langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(\xi_i+k_i)^2\left(\langle k \rangle^{-\frac{2\alpha_o}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o-4}{2}} \\ &\quad + \alpha_o t\langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}}\left(\langle k \rangle^{\frac{-2\alpha}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o-2}{2}},\end{aligned}$$

if $i = j$. Also note $\dfrac{\partial\Phi}{\partial\xi_i} = 0$ when

$$x_i = -\alpha_o t\langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(\xi_i+k_i)\left(\langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o-2}{2}},$$

or equivalently

$$x = -\alpha_o t\langle k \rangle^{\frac{\alpha_o\alpha}{1-\alpha}}(\xi+k)\left(\langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi+k|^2\right)^{\frac{\alpha_o-2}{2}}.$$

Now for the case of $n = 2$ we have

$$
\begin{aligned}
&\left|\det(D_{\xi_i} D_{\xi_j} \Phi)_{i,j=1}^2\right| \\
&= \Bigg|\Bigg(\alpha_o(\alpha_o - 2)t \left\langle k \right\rangle^{\frac{\alpha_o \alpha}{1-\alpha}} (\xi_1 + k_1)^2 \left(\left\langle k \right\rangle^{-\frac{2\alpha_o}{1-\alpha}} + |\xi + k|^2\right)^{\frac{\alpha_o - 4}{2}} \\
&\quad +\alpha_o t \left\langle k \right\rangle^{\frac{\alpha_o \alpha}{1-\alpha}} \left(\left\langle k \right\rangle^{\frac{-2\alpha}{1-\alpha}} + |\xi + k|^2\right)^{\frac{\alpha_o - 2}{2}}\Bigg) \\
&\quad \times \Bigg(\alpha_o(\alpha_o - 2)t \left\langle k \right\rangle^{\frac{\alpha_o \alpha}{1-\alpha}} (\xi_2 + k_2)^2 \left(\left\langle k \right\rangle^{-\frac{2\alpha_o}{1-\alpha}} + |\xi + k|^2\right)^{\frac{\alpha_o - 4}{2}} \\
&\quad +\alpha_o t \left\langle k \right\rangle^{\frac{\alpha_o \alpha}{1-\alpha}} \left(\left\langle k \right\rangle^{\frac{-2\alpha}{1-\alpha}} + |\xi + k|^2\right)^{\frac{\alpha_o - 2}{2}}\Bigg) \\
&\quad -\alpha_o^2(\alpha_o - 2)^2 t^2 \left\langle k \right\rangle^{\frac{2\alpha_o \alpha}{1-\alpha}} (\xi_1 + k_1)^2 (\xi_2 - k_2)^2 \left(\left\langle k \right\rangle^{-\frac{2\alpha}{1-\alpha}} - |\xi + k|^2\right)^{\alpha_o - 4}\Bigg| \\
&= \alpha_0^2 t^2 \left\langle k \right\rangle^{\frac{2\alpha_o \alpha}{1-\alpha}} \left|\left\langle k \right\rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2\right|^{\frac{2\alpha_o - 4}{2}} \\
&\qquad \times \left((\alpha_o - 2)\left(\left\langle k \right\rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2\right)^{-1} |\xi + k|^2 + 1\right).
\end{aligned}
$$

Then $\left|\det(D_{\xi_i} D_{\xi_j} \Phi)_{i,j=1}^2\right| = 0$ only if

$$
(\alpha_o - 2)\left(\left\langle k \right\rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2\right)^{-1} |\xi + k|^2 + 1 = 0,
$$

which only happens when

$$
\alpha_o = 1 - \left\langle k \right\rangle^{-\frac{2\alpha}{1-\alpha}} |\xi + k|^{-2} < 1.
$$

Thus when $\alpha_o \geq 1$ and when $k \neq 0$, $\left|\det(D_{\xi_i} D_{\xi_j} \Phi)_{i,j=1}^2\right| \neq 0$. Also note that when $k \neq 0$

$$
\begin{aligned}
&\left|\det(D_{\xi_i} D_{\xi_j} \Phi)_{i,j=1}^{2}\right| \\
&= \alpha_0^2 t^2 \langle k \rangle^{\frac{2\alpha_o \alpha}{1-\alpha}} \left| \langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2 \right|^{\frac{2\alpha_o - 4}{4}} \\
&\qquad \times \left((\alpha_o - 2) \left(\langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2 \right)^{-1} |\xi + k|^2 + 1 \right) \\
&\sim \alpha_0^2 t^2 \langle k \rangle^{\frac{2\alpha_o \alpha}{1-\alpha}} |\xi + k|^{2\alpha_o - 4} \left(\alpha - 1 + \frac{1}{1 + \langle k \rangle^{\frac{2\alpha}{1-\alpha}} |\xi + k|^2} \right) \\
&\geq \left(\alpha_0 t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |\xi + k|^{\alpha_o - 2} \right)^2 \left(\alpha - 1 + \frac{1}{1 + \langle k \rangle^{\frac{2\alpha}{1-\alpha}} |\xi + k|^2} \right) \\
&\geq \alpha_0 t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2}.
\end{aligned}
$$

Define $C_i(k)$ and $D_i(k)$ as

$$
\begin{aligned}
C_i(k) &= \alpha_o t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} (|k_i| + C) \left(\sum_{j=1}^{n} (|k_j| + C)^2 \right)^{\frac{\alpha_o - 2}{2}}, \\
D_i(k) &= \alpha_o t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} (|k_i| - C) \left(\sum_{j=1}^{n} (|k_j| - C)^2 \right)^{\frac{\alpha_o - 2}{2}}.
\end{aligned}
$$

Now define the intervals F_i as the set of all $x_i \in \mathbb{R}$, such that,

$$
D_i(k) - t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2} < |x_i| < C_i(k) + t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2},
$$

$G_{i,j}$ to be the set of all $x_i \in \mathbb{R}$ such that

$$
C_i(k) + t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2} + j - 1 < |x_i| \leq C_i(k) + t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2} + j,
$$

and $H_{i,j}$ to be the set of all $x_i \in \mathbb{R}$ such that

$$
D_i(k) - t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2} - j < |x_i| \leq D_i(k) + t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2} - j + 1.
$$

Since

$$
|x| = \alpha_o t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |\xi + k| \left| \langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2 \right| |^{\frac{\alpha_o - 2}{2}},
$$

then it follows that

$$x_i \in F_i.$$

Also, it follows that

$$\text{length}(F_i) \preceq t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2},$$
$$\text{length}(G_{i,j}) = \text{length}(H_{i,j}) = 1.$$

Now define $K_{i,j}$ as

$$K_{i,j} = G_{i,j} \cup H_{i,j},$$

then it follows that

$$\chi_{F_i}(x_i) + \sum_{j=1}^{\infty} \chi_{K_{i,j}}(x_i) = 1.$$

Thus we have

$$\begin{aligned}
&\left|\left|\mathscr{F}^{-1}(\eta_k^\alpha(\xi) e^{it(\xi)^{\alpha_o}})\right|\right|_{L^1} \preceq \\
&\int_{\mathbb{R}^n} \prod_{i=1}^{n} \chi_{F_i}(x_i) \left| \int_{\mathbb{R}^n} \eta_k^\alpha\left(\xi_k^\alpha\right) e^{it(\xi)_k^{\alpha,\alpha_o} + ix\xi} d\xi \right| dx + \\
&\quad \sum_{j_*=1}^{n} \sum_{I_l} \int_{\mathbb{R}^n} \mathscr{A}_{I_l}(x) \left| \int_{\mathbb{R}^n} \eta_k^\alpha\left(\xi_k^\alpha\right) e^{it(\xi)_k^{\alpha,\alpha_o} + ix\xi} d\xi \right| dx + \\
&\quad \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \chi_{K_{i,j_i}}(x_i) \left| \int_{\mathbb{R}^n} \eta_k^\alpha\left(\xi_k^\alpha\right) e^{it(\xi)_k^{\alpha,\alpha_o} + ix\xi} d\xi \right| dx \\
&I_1 + I_2 + I_3,
\end{aligned}$$

where $\mathscr{A}_{I_l}$ is the product of characteristic functions $\chi_{F_i}(x_i)$ and $\chi_{K_{i,j_*}}(x_i)$ where there is at least one $\chi_{F_i}(x_i)$ and at least one $\chi_{K_{i,j_*}}(x_i)$.

For I_1, with $\xi \in \operatorname{supp} \eta_k^\alpha\left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k)\right)$ and by Lemma 3.2.4 we have

$$\begin{aligned}
I_1 &\preceq \left(t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2}\right)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \chi_{F_i}(x_i) dx \\
&\preceq \left(t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2}\right)^{-\frac{n}{2}} \left(t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2}\right)^{n} \\
&= t^{\frac{n}{2}} \langle k \rangle^{\frac{n \alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{n(\alpha_o - 2)}{2}} \\
&\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\alpha_o - 2 + 2\alpha)}{2(1-\alpha)}}.
\end{aligned}$$

Now note for $x \in K_{i,j}$ and $\xi \in \text{supp}\eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right)$ we have

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_l} \left(\frac{\eta_k^\alpha (\xi_k^\alpha)}{\frac{\partial}{\partial \xi_i} \Phi} \right) \\
&= \frac{\frac{\partial \Phi}{\partial \xi_i} \frac{\partial}{\partial \xi_l} \eta_k^\alpha (\xi_k^\alpha) - \eta_k^\alpha (\xi_k^\alpha) \frac{\partial^2 \Phi}{\partial \xi_l \partial \xi_i}}{\left(\frac{\partial \Phi}{\partial \xi_i} \right)^2} \\
&\preceq \frac{\alpha_o t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |\xi + k|^{\alpha_o - 2} (\xi_i + k_i) + x_i - \frac{\partial^2 \Phi}{\partial \xi_l \partial \xi_i}}{\left(\alpha_o t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} (\xi_i + k_i) \left| \langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2 \right|^{\frac{\alpha_o - 2}{2}} + x_i \right)^2} \\
&= O \left(\frac{1}{j + \sqrt{t} \langle k \rangle^{\frac{\alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{\alpha_o - 2}{2}}} + \frac{t \langle k \rangle^{\frac{\alpha_o \alpha}{1-\alpha}} |k|^{\alpha_o - 2}}{\left(j + \sqrt{t} \langle k \rangle^{\frac{\alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{\alpha_o - 2}{2}} \right)^2} \right).
\end{aligned}
$$

Thus, using integration-by-parts twice on each variable $\xi_1, \cdots, \xi_n$ we have

$$
\begin{aligned}
I_3 &\preceq \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{t^n \langle k \rangle^{\frac{n\alpha_o \alpha}{1-\alpha}} |k|^{n\alpha_o - 2n}}{\prod_{i=1}^n \left(j_i + \sqrt{t} \langle k \rangle^{\frac{\alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{\alpha_o - 2}{2}} \right)^2} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \chi_{K_{i,j_i}}(x_i) dx \\
&\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{n(\alpha_o - 2)}{2}} \\
&\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\alpha_o - 2 + 2\alpha)}{2(1-\alpha)}}.
\end{aligned}
$$

When $\xi \in \text{supp}\, \eta_k^\alpha (\xi_k^\alpha)$, then I_2 is the sum of integrals of the form

$$
\begin{aligned}
\sum_{j_{l+1}=1}^{n} \cdots \sum_{j_n=1}^{n} \int_{\mathbb{R}^n} \prod_{i_0=1}^{l} \chi_{F_{i_o}}(x_{i_0}) \prod_{i_0=l+1}^{n} \chi_{K_{i_0, j_{i_0}}}(x_{i_0}) \times \\
\left| \int_{\mathbb{R}^n} \eta_k^\alpha (\xi_k^\alpha) e^{it(\xi)_k^{\alpha,\alpha_o} + x\xi} d\xi \right| dx.
\end{aligned}
$$

So doing integration-by-parts twice on the variables $\xi_{l+1}, \cdots, \xi_n$, the above integral is bounded by

$$\sum_{j_{l+1}=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{t^{n-l} \langle k \rangle^{\frac{(n-l)\alpha_o \alpha}{1-\alpha}} |k|^{(n-l)\alpha_o - 2n}}{\prod_{i=l+1}^{n} \left(j_i + \sqrt{t} \langle k \rangle^{\frac{\alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{\alpha_o - 2}{2}} \right)^2}$$

$$\times \int_{\mathbb{R}^n} \prod_{i_0=1}^{l} \chi_{F_{i_o}}(x_{i_0}) \prod_{i_0=l+1}^{n} \chi_{K_{i_0, j_{i_0}}}(x_{i_0}) dx$$

$$\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\alpha_o \alpha}{2(1-\alpha)}} |k|^{\frac{n(\alpha_o - 2)}{2}}$$

$$\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\alpha_o - 2 + 2\alpha)}{2(1-\alpha)}} .$$

This completes the proof. □

Applying both Proposition 3.4.2 and 3.4.3 with Proposition 3.2.6 from Section 3.2 we obtain the proof for Theorem 3.1.3

3.5 Proof for the Asymptotic Estimate for the Fourier Multiplier $\Theta(t)$

Now we will present the proof for Theorem 3.1.4.

Proof. First note that by Plancheral's Theorem, Theorem 1.2.1, it follows

$$\begin{aligned} ||\square_k^\alpha \Theta(t) g||_{L^2} &= \left|\left| \mathscr{F}^{-1} \left(\eta_k^\alpha(\xi) \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi) \right) \right|\right|_{L^2} \\ &= \left|\left| \eta_k^\alpha(\xi) \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi) \right|\right|_{L^2} \\ &\preceq ||\hat{g}||_{L^2} \\ &= ||g||_{L^2} . \end{aligned}$$

Now when $k \neq 0$ such that $|k| > C$, then $|\xi + k| = 0$ is not in the support of $\eta_k^\alpha(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k))$. With L defined as

$$L = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Taking multiple derivatives of $\sin\left(t\langle k\rangle^{\frac{\alpha}{1-\alpha}}|\xi+k|\right)$ we get lead factors of t^L and $\langle k\rangle^{\frac{L\alpha}{1-\alpha}}$ and with k sufficiently large we have for all the factors of the form $|\xi+k|^{-j}$ where $j\in\mathbb{N}$ is dominated by $|\xi+k|^{-1}$. Since $|\xi+k|^{-1}\preceq\langle k\rangle^{-1}$ when $|k|\geq 1$ we get the following

$$\begin{aligned}
&||\Box_k^\alpha\Theta(t)g||_{L^1}\\
&\preceq\left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k)}{\langle k\rangle^{\frac{\alpha}{1-\alpha}}|\xi+k|}\right|\right|_{L^2}^{1-\frac{n}{2L}}\\
&\qquad\times\sum_{|\delta|=L}\left|\left|D^\delta\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))\sin\left(t\langle k\rangle^{\frac{\alpha}{1-\alpha}}|\xi+k|\right)}{\langle k\rangle^{\frac{\alpha}{1-\alpha}}|\xi+k|}\right|\right|_{L^2}^{\frac{n}{2L}}||g||_{L^1}\\
&\preceq t^{\frac{n}{2}}\langle k\rangle^{-1+\frac{n}{2L}}\langle k\rangle^{-\frac{\alpha}{1-\alpha}+\frac{n\alpha}{2L(1-\alpha)}}\langle k\rangle^{-\frac{n}{2L}}\langle k\rangle^{\frac{n\alpha}{2(1-\alpha)}-\frac{n\alpha}{2L(1-\alpha)}}||g||_{L^1}\\
&=t^{\frac{n}{2}}\langle k\rangle^{-\frac{\alpha}{(1-\alpha)}}\langle k\rangle^{\frac{\alpha(n-2)}{2(1-\alpha)}}||g||_{L^1}\\
&=t^{\frac{n}{2}}\langle k\rangle^{\frac{\alpha n-2}{2(1-\alpha)}}||g||_{L^1}.
\end{aligned}$$

Now by Riesz-Thorin Interpolation, proposition 3.2.5, with $1\leq p\leq 2$, which will satisfy the following

$$\frac{1}{p}=\frac{1-\theta}{2}+\theta,$$

which implies that

$$\theta=2\left(\frac{1}{p}-\frac{1}{2}\right),$$

we have the following estimate

$$\begin{aligned}
||\Box_k^\alpha\Theta(t)g||_{L^p}&\preceq\left(t^{\frac{n}{2}}\langle k\rangle^{\frac{\alpha n-2}{2(1-\alpha)}}\right)^\theta(||g||_{L^2})^{1-\theta}||g||_{L^p}\\
&=\langle k\rangle^{t^{\frac{\theta n}{2}}\frac{(\alpha n-2)\theta}{2(1-\alpha)}}||g||_{L^2}^\theta||g||_{L^2}\\
&\preceq t^{n\left(\frac{1}{p}-\frac{1}{2}\right)}\langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left(\frac{1}{p}-\frac{1}{2}\right)}||g||_{L^p}.
\end{aligned}$$

Now by a dual argument we have

$$||\Box_k^\alpha\Theta(t)g||_{L^p}\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|}||g||_{L^p}.$$

for $1\leq p\leq\infty$.

Now by almost orthogonality it follows that

$$\Box_k^\alpha \Theta(t) = \sum_{|l| \leq \gamma_{C,k}} \Box_{k+l}^\alpha \Box_k^\alpha \Theta(t) g,$$

for some constant $\gamma_{C,k} > 0$ that depends on C and k. Then by definition of $\Box_k^\alpha$ it follows

$$\begin{aligned} ||\Box_k^\alpha \Theta(t) g||_{L^p} &= \left|\left| \sum_{|l| \leq \gamma_{C,k}} \Box_{k+l}^\alpha \Box_k^\alpha \Theta(t) g \right|\right|_{L^p} \\ &= \left|\left| \sum_{|l| \leq \gamma_{C,k}} \Box_{k+l}^\alpha \Theta(t) \Box_k^\alpha g \right|\right|_{L^p} \\ &\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k \rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{L^p} ||\Box_k^\alpha g||_{L^p}, \end{aligned}$$

when $k \neq 0$ and $1 \leq p \leq \infty$.

Now for $k = 0$, by Lemma 3.2.8 we have

$$||\Box_0^\alpha \Theta(t) g||_{L^p} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{L^p},$$

or

$$||\Box_k^\alpha \Theta(t) g||_{L^p} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{L^p},$$

for small k.

Now by the definition of the α-modulation space it follows that

$$
\begin{aligned}
&||\Theta(t)g||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \\
&= \left(\sum_{k\in\mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} ||\Box_k^\alpha \Theta(t) g||_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\
&\preceq \left(\sum_{|k|<N} \langle k \rangle^{\frac{sq}{1-\alpha}} ||\Box_k^\alpha \Theta(t) g||_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} + \\
&\qquad \left(\sum_{|k|\geq N} \langle k \rangle^{\frac{sq}{1-\alpha}} ||\Box_k^\alpha \Theta(t) g||_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\
&t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \left(\sum_{|k|<N} \langle k \rangle^{\frac{sq}{1-\alpha}} ||\Box_k^\alpha g||_{L^p(\mathbb{R}^n(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} + \\
&\qquad t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|\geq N} \langle k \rangle^{\frac{sq}{1-\alpha}} \langle k \rangle^{\frac{q(\alpha n-2)}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||\Box_k^\alpha g||_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\
&\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \left(\sum_{|k|<N} ||\Box_k^\alpha g||_{L^p(\mathbb{R}^n(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} + \\
&\qquad t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|\geq N} \langle k \rangle^{\frac{q}{1-\alpha}\left(s+(\alpha n-2)\left|\frac{1}{p}-\frac{1}{2}\right|\right)} ||\Box_k^\alpha g||_{L^p(\mathbb{R}^n(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\
&\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M_{p,q}^{s+\beta_1(\alpha),\alpha}(\mathbb{R}^n)},
\end{aligned}
$$

where $\gamma \geq 0$ and $\beta_1(\alpha)$ is defined as

$$\beta_1(\alpha) = (\alpha n - 2)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

□

3.6 Proof for the Asymptotic Estimate for the Fourier Multiplier $\Theta_K(t)$

Now we will present the proof for Theorem 3.1.5 but first we need two propositions:

Proposition 3.6.1. *(Trulen) For $1 \leq p, q \leq \infty$, $t \geq 1$ and $|k| = 0$, then we have the following estimate:*

$$||\Box_k^\alpha \Theta_K(t) g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{L^p(\mathbb{R}^n)} .$$

Proof. Suppose $|k| = 0$. Let L be defined by

$$L = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n+2}{2}, & \text{if } n \text{ is even,} \end{cases}$$

then by Bernsteins Multiplier Theorem, proposition 3.2.2, we have

$$\begin{aligned} &||\Box_0^\alpha \Theta_K(t) g||_{L^1(\mathbb{R}^n)} \\ &\preceq \left|\left|\eta_0^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \\ &\quad \times \sum_{|\delta|=L} \left|\left| D^\delta \left(\eta_0^k(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right)\right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} ||g||_{L^1(\mathbb{R}^n)} . \end{aligned}$$

For the first norm we have

$$\begin{aligned} \left|\left|\eta_0^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} &= \left|\left| t\, \eta_0^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{t(1+|\xi|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \\ &\preceq t^{1-\frac{n}{2L}} . \end{aligned}$$

For the second norm, define h as

$$h(|\xi|) = \frac{\sin((1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} .$$

By the Taylor expansion of sine we have

$$h(|\xi|) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(1+|\xi|^2)^k}{(2k+1)!} ,$$

and thus h is a C^∞ function.

Also, by doing copious amounts of derivatives we have

$$\lim_{|\xi|\to\infty} \left|D^\delta h(|\xi|)\right| = 0,$$

for all multi-indeces δ. Noticing that

$$th(t|\xi|) = \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}},$$

then when $|\delta| = L$ we have

$$\begin{aligned} \left| D^\delta \left(\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right) \right| &\preceq t^{L+1} \sup_{\xi\in\mathbb{R}^n} \left|D^\delta h(|\xi|)\right| \\ &\preceq t^{L+1}. \end{aligned}$$

Now it follows that

$$\begin{aligned} \sum_{|\delta|=L} \left|\left| D^\delta \left(\eta_0^k(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right) \right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} &\preceq \left(t^{L+1}\right)^{\frac{n}{2L}} \\ &= t^{\frac{n}{2}} t^{\frac{n}{2L}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} ||\Box_0^\alpha \Theta_K(t) g||_{L^1(\mathbb{R}^n)} &\preceq t^{\frac{n}{2}} t^{\frac{n}{2L}} t^{1-\frac{n}{2L}} \, ||g||_{L^1(\mathbb{R}^n)} \\ &= t^{\frac{n+2}{2}} \, ||g||_{L^1(\mathbb{R}^n)} \,. \end{aligned}$$

By Plancherel's Theorem, Theorem 1.2.1, it follows that

$$\begin{aligned}
||\Box_0^\alpha \Theta_K(t)g||_{L^2(\mathbb{R}^n)} &= \left|\left|\mathscr{F}^{-1}\left(\eta_0^\alpha(\xi)\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}\hat{g}(\xi)\right)\right|\right|_{L^2(\mathbb{R}^n)} \\
&= \left|\left|\eta_0^\alpha(\xi)\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}\hat{g}\right|\right|_{L^2(\mathbb{R}^n)} \\
&\preceq \left|\left|\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}\hat{g}\right|\right|_{L^2(\mathbb{R}^n)} \\
&= \left|\left|t\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{t(1+|\xi|^2)^{\frac{1}{2}}}\hat{g}\right|\right|_{L^2(\mathbb{R}^n)} \\
&\preceq t\,||\hat{g}||_{L^2(\mathbb{R}^n)} \\
&= t\,||g||_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Now by Riesz-Thorin Interpolation, Proposition 3.2.5, and a duality argument it follows that

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\,||g||_{L^p(\mathbb{R}^n)}.$$

This ends the proof. $\square$

Proposition 3.6.2. *(Trulen) For $1 \leq p, q \leq \infty$, $t \geq 1$, and $|k| \neq 0$, then we have the following estimate:*

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\,\langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|}\,||g||_{L^p(\mathbb{R}^n)}.$$

Proof. Again define L to be

$$L = \begin{cases} \dfrac{L+1}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{L+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Then by the usual substitutions and Bernsteins Multiplier Theorem, Proposition

3.2.2, we have

$$\begin{aligned}
&\left|\left|\Box_k^\alpha \Theta_K(t) g\right|\right|_{L^1(\mathbb{R}^n)} \preceq \\
&\left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \\
&\qquad \times \sum_{|\delta|=L}\left|\left|D^\delta \frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))\sin(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}})}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} \\
&\qquad \times ||g||_{L^1(\mathbb{R}^n)}\,.
\end{aligned}$$

For the first norm, noting that for large enough k we have $(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2)^{-\frac{1}{2}} \preceq \langle k\rangle^{-1}$ it follows that

$$\begin{aligned}
&\left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \\
&= \langle k\rangle^{-\frac{\alpha}{1-\alpha}+\frac{n}{2L}}\left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))}{(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \\
&\preceq \langle k\rangle^{-\frac{\alpha}{1-\alpha}+\frac{n}{2L}}\langle k\rangle^{-1+\frac{n}{2L}}\,.
\end{aligned}$$

For the second norm, $D^\delta \sin(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}})$ produces t^L and $\langle k\rangle^{\frac{L\alpha}{1-\alpha}}$ factors when $|\delta|=L$. Also, after taking multiple derivatives we have the remaining factors of the form $(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{-\frac{j}{2}}$ for some positive integer j which again for large enough k we have

$$\begin{aligned}
(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{-\frac{j}{2}} &\preceq \langle k\rangle^{-\frac{j\alpha}{1-\alpha}}\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{-\frac{j}{2}} \\
&\preceq \langle k\rangle^{-\frac{\alpha}{1-\alpha}}\langle k\rangle^{-1}\,.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\sum_{|\delta|=L}\left|\left|D^\delta \frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))\sin(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}})}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} ||g||_{L^1(\mathbb{R}^n)} \\
&\preceq t^{\frac{n}{2}}\langle k\rangle^{\frac{n\alpha}{2(1-\alpha)}}\langle k\rangle^{-\frac{n\alpha}{2L(1-\alpha)}}\langle k\rangle^{-\frac{n}{2L}}||g||_{L^1(\mathbb{R}^n)}\,.
\end{aligned}$$

Then it follows that

$$
\begin{aligned}
&||\square_k^\alpha \Theta_K(t) g||_{L^1(\mathbb{R}^n)}\\
&\preceq t^{\frac{n}{2}} \langle k\rangle^{-\frac{\alpha}{1-\alpha}+\frac{n}{2L}} \langle k\rangle^{-1+\frac{n}{2L}} \langle k\rangle^{\frac{n\alpha}{2(1-\alpha)}} \langle k\rangle^{-\frac{n\alpha}{2L(1-\alpha)}} \langle k\rangle^{-\frac{n}{2L}} ||g||_{L^1(\mathbb{R}^n)}\\
&= t^{\frac{n}{2}} \langle k\rangle^{\frac{\alpha n-2}{2(1-\alpha)}} ||g||_{L^1(\mathbb{R}^n)} .
\end{aligned}
$$

Using Plancherel's Theorem, Theorem 1.2.1, we have

$$
\begin{aligned}
||\square_k^\alpha \Theta_K(t) g||_{L^2(\mathbb{R}^n)} &= \left|\left|\eta_k^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \hat{g}\right|\right|_{L^2(\mathbb{R}^n)}\\
&\preceq ||\hat{g}||_{L^2(\mathbb{R}^n)}\\
&= ||g||_{L^2(\mathbb{R}^n)} .
\end{aligned}
$$

By Riesz-Thorin and a duality argument it follows that for $1 \leq p \leq \infty$

$$
||\square_k^\alpha \Theta_K(t) g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{L^p(\mathbb{R}^n)} ,
$$

and this finishes the proof. □

To obtain Theorem 3.1.5 we need a simple almost orthogonality argument and the definition of α-modulation space $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$.

Proof. By almost orthogonality we have

$$
\begin{aligned}
\square_k^\alpha \Theta_K(t) g &= \sum_{|l|\leq \gamma_{C,k}} \square_{k+l}^\alpha \square_k^\alpha \Theta_K(t) g\\
&= \sum_{|l|\leq \gamma_{C,k}} \square_{k+l}^\alpha \Theta_K(t) \square_k^\alpha g.
\end{aligned}
$$

Then by the Proposition 3.6.1 and when $k = 0$ we have

$$
||\square_k^\alpha \Theta_K(t) g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||\square_k^\alpha g||_{L^p(\mathbb{R}^n)} ,
$$

and by the Proposition 3.6.2 and when $k \neq 0$ we have

$$
||\square_k^\alpha \Theta_K(t) g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||\square_k^\alpha g||_{L^p(\mathbb{R}^n)} .
$$

By definition of the α-modulation norm we have

$$
\begin{aligned}
||\Theta_K(t)g||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} &= \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\Box^\alpha_k \Theta_K(t)g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\preceq \left(\sum_{|k|<N} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\Box^\alpha_k \Theta_K(t)g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} + \\
&\quad \left(\sum_{|k|\geq N} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\Box^\alpha_k \Theta_K(t)g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \left(\sum_{|k|<N} \langle k\rangle^{\frac{sq}{1-\alpha}} ||\Box^\alpha_k g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} + \\
&\quad t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|\geq N} \langle k\rangle^{\frac{sq}{1-\alpha}+\frac{q(n\alpha-2)}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||\Box^\alpha_k g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \left(\sum_{|k|<N} ||\Box^\alpha_k g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} + \\
&\quad t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(\sum_{|k|\geq N} \langle k\rangle^{\frac{q}{1-\alpha}\left(s+(n\alpha-2)\left|\frac{1}{p}-\frac{1}{2}\right|\right)} ||\Box^\alpha_k g||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M^{s+\beta_1(\alpha),\alpha}_{p,q}(\mathbb{R}^n)}.
\end{aligned}
$$

This completes the proof. □

3.7 Asymptotic Estimates for Homogeneous Solutions

Now we will state the asymptotic behavior of the Schrödinger, Airy, wave, and Klein-Gordon equations. The first statement deals with the Schrödinger equation.

Corollary 3.7.1. *(Trulen) Let* $1 \leq p, q \leq \infty$, $t \geq 1$, *and* $u(t,x)$ *be the solution*

Cauchy problem for the Schrödinger equation

$$\begin{cases} \partial_t u = i\Delta_x u, & \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f(x), & \textit{for } x \in \mathbb{R}^n, \end{cases}$$

then

$$\left|\left|e^{it|\Delta|}f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq ||f||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+\beta_2(\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\beta_2(\alpha)$ *is*

$$\beta_2(\alpha) = 2n\alpha\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Proof. The formal solution to this equation is given by

$$u(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{it|\xi|^2}\hat{f}(\xi)e^{ix\xi} = (e^{-it\Delta}f)(x).$$

Then by Theorem 3.1.2, we achieve the desired result. □

Corollary 3.7.2. *(Trulen) Let* $1 \leq p,q \leq \infty$*,* $t \geq 1$ *and* $u(t,x)$ *be the solution of Cauchy problem for the wave equation*

$$\begin{cases} \partial_{tt}u(t,x) = \Delta u(t,x), & \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f(x), & \textit{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = g(x), & \textit{for } x \in \mathbb{R}^n, \end{cases}$$

then we have the following estimate:

$$||u(t,x)||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s-\gamma_1,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+\beta(\alpha),\alpha}_{p,q}(\mathbb{R}^n)} +$$
$$t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M^{s-\gamma_2,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M^{s+\beta_1(\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma_1, \gamma_2 \geq 0$*,* $\beta(\alpha)$ *is defined by equation* (3.1)*, and* $\beta_1(\alpha)$ *is defined by equation* (3.3)*.*

Proof. The formal solution to the wave equation is given by

$$u(t,x) = \cos(t(-\Delta))f(x) + \frac{\sin(t(-\Delta))}{(-\Delta)}g(x).$$

Then from Theorems 3.1.1 and 3.1.4 the result follows. □

Next we state a similar estimate for the Airy equation.

Corollary 3.7.3. *(Trulen) Let $1 \leq p, q \leq \infty$, $t \geq 1$ and $u(t,x)$ be the solution to the Cauchy problem for the Airy equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} = |\Delta_x|^{\frac{3}{2}} u, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \\ u(0,x) = f(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

then

$$\left|\left|(e^{it|\Delta|^{\frac{3}{2}}} f)(x)\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq ||f||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+\beta_3(\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma \geq 0$ and $\beta_3(\alpha)$ is defined as

$$\beta_3(\alpha) = n(2\alpha+1)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Proof. The formal solution to the above equation is given by

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|^3} \hat{f}(\xi) e^{ix\xi} d\xi \\ &= (e^{it|\Delta|^{\frac{3}{2}}} f)(x). \end{aligned}$$

Again, by theorem 3.1.2 we achieve the desired result. ∎

Corollary 3.7.4. *(Trulen) Let $1 \leq p, q \leq \infty$, $t \geq 1$, and $u(t,x)$ be the solution to the Cauchy Problem for the Klein-Gordon Equation*

$$\begin{cases} \partial_{tt} u(t,x) + u(t,x) - \Delta u(t,x) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f(x), & \text{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = g(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

then we have the followings estimate

$$\begin{aligned} ||u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq{}& t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s-\gamma_1,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+\beta(1,\alpha),\alpha}_{p,q}(\mathbb{R}^n)} \\ &+ t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M^{s-\gamma_2,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M^{s+\beta_1(\alpha),\alpha}_{p,q}(\mathbb{R}^n)}, \end{aligned}$$

where γ_1 and γ_2 are positive real numbers, $\beta(1,\alpha)$ and $\beta_1(\alpha)$ are defined by

$$\beta(1,\alpha) = n(2\alpha-1)\left|\frac{1}{p} - \frac{1}{2}\right|, \text{ and } \beta_1(\alpha) = (n\alpha - 2)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Proof. The formal solution to the Klein-Gordon equation is given by

$$u(t,x) = \cos(t(I-\Delta)^{\frac{1}{2}})f(x) + \frac{\sin(t(I-\Delta)^{\frac{1}{2}})}{(I-\Delta)}g(x).$$

Then from theorems 3.1.3 and 3.1.5 the result follows. □

Chapter 4

Nonlinear Dispersive Equations

4.1 Solution to the Nonlinear Cauchy Problem for Dispersive Equations

Han and Wang [18] or Chapter 2 Section 2.3, Proposition 2.3.13, state conditions on p, q, and s that makes $M_{p,q}^{s,\alpha}$ a multiplication algebra; i.e. for any $f, g \in M_{p,q}^{s,\alpha}$

$$||fg||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq ||f||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \, ||g||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \, .$$

Since Theorems 3.1.1 and 3.1.2 are for values of $1 \leq p, q \leq \infty$ this changes the region D_1 to

$$D_1 = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \mathbb{R}_+^2 : \frac{2}{p} \leq \frac{1}{q} \leq 1 \right\},$$

and D_2 changes to

$$D_2 = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in \mathbb{R}_+^2 \backslash D_1 : \frac{1}{p}, \frac{1}{q} \leq 1 \right\}.$$

See figure 4.1 for the new distribution of s_0. This changes the restriction on s_0 to equal

$$s_0 = \frac{n\alpha}{p} + n(1-\alpha)\left(1 - \frac{1}{q}\right) + \frac{n\alpha(1-\alpha)}{2-\alpha}\left(\frac{1}{q} - \frac{2}{p}\right),$$

when $\left(\frac{1}{p}, \frac{1}{q} \right) \in D_1$, and

$$s_0 = \frac{n\alpha}{p} + n(1-\alpha)\left(1 - \frac{1}{q}\right),$$

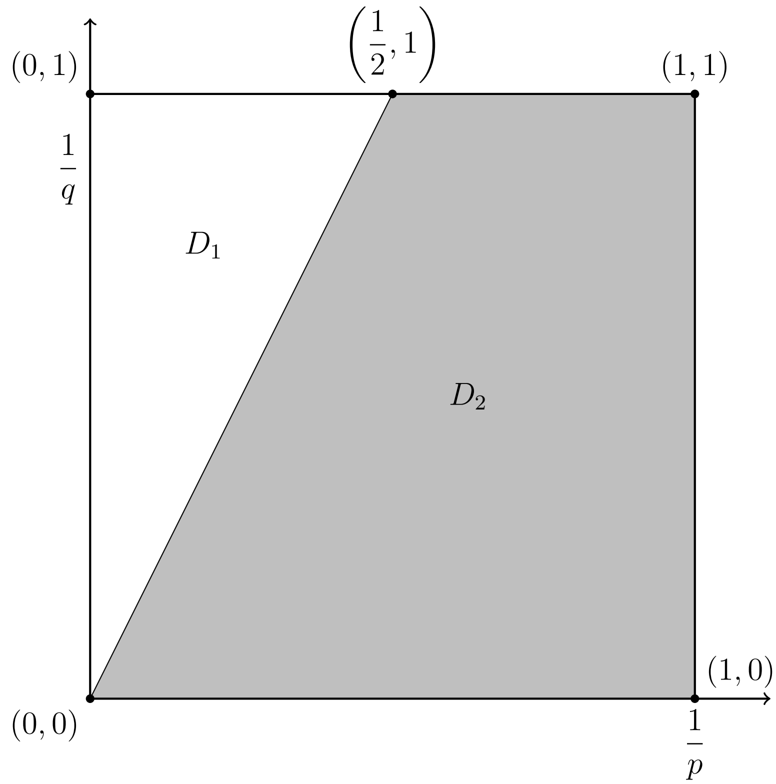

Figure 4.1: Distribution of s_0 when $p \geq 1$ and $q \geq 1$.

when $\left(\frac{1}{p}, \frac{1}{q}\right) \in D_2$.

Define the function space $C\left([0,T], M^{s,\alpha}_{p,q}\right)$ by

$$C\left([0,T], M^{s,\alpha}_{p,q}\right) = \left\{u(t,x) : ||u||_{C\left([0,T], M^{s,\alpha}_{p,q}\right)} < \infty\right\},$$

where $||u||_{C\left([0,T], M^{s,\alpha}_{p,q}\right)}$ is defined as

$$||u||_{C\left([0,T], M^{s,\alpha}_{p,q}\right)} = \sup_{0 \leq t \leq T} ||u(t,\cdot)||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.$$

Before we can prove our result we need the following lemma obtained from Fan, Chen, and Sun [5].

Lemma 4.1.1. *([5]) The following estimate holds:*

$$\begin{aligned}&\left||u(\tau,\cdot)|^{2k}u(\tau,\cdot)-|v(\tau,\cdot)|^{2k}v(\tau,\cdot)\right|\\&\leq\left(|u-v||u(\tau,\cdot)|^{2k}\right)+\left\{\sum_{j=0}^{k-1}|u|^{j}|v|^{k-j}(|u(\tau,\cdot)|^{k})+|v(\tau,\cdot)|^{k})\right\}.\end{aligned}$$

Now we are able to state our result.

Theorem 4.1.2. *(Trulen) For $1\leq p,q\leq\infty$, $s>s_0$, and $T\geq 1$. Suppose k is a positive integer and there is positive constant c_k dependent only on k such that*

$$||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\leq\frac{c_k}{T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)}T^{\frac{1}{2k}}}.$$

Suppose $\frac{1}{2}<\alpha_o\leq 2(1-\alpha)$ with $\alpha_o\neq 1$, then the nonlinear Cauchy problem for dispersive equation

$$\begin{cases} i\partial_t u-|\Delta|^{\frac{\alpha_o}{2}}u+F(u)=0, & \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\\ u(0,x)=u_0(x), & \text{for } x\in\mathbb{R}^n,\end{cases}$$

when $F(u)=|u|^{2k}u$ has a unique solution $u\in C([0,T],M^{s,\alpha}_{p,q})$.

Proof. The Nonlinear Cauchy problem for disperive equation has an equivalent form

$$u(t,\cdot)=e^{it|\Delta|^{\frac{\alpha_o}{2}}}u_0-i\int_0^t e^{i(t-\tau)|\Delta|^{\frac{\alpha_o}{2}}}F(u(\tau,\cdot))d\tau.$$

Consider the mapping

$$\mathscr{T}u=e^{it|\Delta|^{\frac{\alpha_o}{2}}}u_0-i\int_0^t e^{i(t-\tau)|\Delta|^{\frac{\alpha_o}{2}}}F(u(\tau,\cdot))d\tau.$$

Let C_j where $j=1,2,3$ denote some positive constants that are independent of all essential variables. By Theorems 3.1.1 and 3.1.2 we have

$$\left|\left|e^{it|\Delta|^{\frac{\alpha_o}{2}}}u_0\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\leq C_1(1+t)^{n\left|\frac{1}{p}-\frac{1}{2}\right|}||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.$$

By Proposition 2.3.13 there is a constant $A_{2k+1}>0$ for $s>s_0$ such that

$$\left|\left||u(t,\cdot)|^{2k+1}\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\leq A_{2k+1}||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.$$

Let M_k be defined as

$$M_k = \max\{A_{2k}, A_{2k+1}\}.$$

Now for any $T \geq 1$ and $t \leq T$

$$\begin{aligned}
&\left|\left|\int_0^t e^{i(t-\tau)|\Delta|^{\frac{\alpha_o}{2}}} F(u(\tau,\cdot))d\tau\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\
&\leq C_1 \int_0^t (1+(t-\tau))^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left||u(\tau,\cdot)|^{2k}u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} d\tau \\
&\leq C_2 M_k T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \sup_{0\leq t\leq T} ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
||\mathscr{T}u||_{C([0,T],M^{s,\alpha}_{p,q})} &= \sup_{0\leq t\leq T} ||\mathscr{T}u(t,\cdot)||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\
&\preceq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T \sup_{0\leq t\leq T} ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right).
\end{aligned}$$

Now let $\mathcal{L}$ be defined as

$$\mathcal{L} = \frac{1}{\left(2C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\right)^{\frac{1}{2k}} (2k+1)^{\frac{1}{2k}}},$$

and let $B_{\mathcal{L}}$ be the closed ball of radius $\mathcal{L}$ centered at the origin in the space of $C([0,T], M^{s,\alpha}_{p,q})$. Suppose that

$$||u_o||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{1}{(2k+1)^{\frac{1}{2k}} (2C_3)^{1+\frac{1}{2k}} T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)} T^{\frac{1}{2k}}}.$$

Thus it follows that

$$\begin{aligned}
||\mathscr{T}u||_{C([0,T],M^{s,\alpha}_{p,q})} &\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T\mathcal{L}^{2k+1}\right) \\
&\leq \mathcal{L},
\end{aligned}$$

and so $\mathscr{T}$ is a mapping from $B_{\mathcal{L}}$ into $B_{\mathcal{L}}$.

Now it follows that

$$
\begin{aligned}
&\mathscr{T}u - \mathscr{T}v \\
&= e^{it|\Delta|^{\frac{\alpha_o}{2}}} u_0 - \int_0^t e^{i(t-\tau)|\Delta|^{\frac{\alpha_o}{2}}} F(u(\tau,\cdot))d\tau \\
&\qquad - e^{it|\Delta|^{\frac{\alpha_o}{2}}} v_0 + \int_0^t e^{i(t-\tau)|\Delta|^{\frac{\alpha_o}{2}}} F(v(\tau,\cdot))d\tau \\
&= e^{it|\Delta|^{\frac{\alpha_o}{2}}} (u_0 - v_0) - \int_0^t e^{i(t-\tau)|\Delta|^{\frac{\alpha_o}{2}}} \left(|u|^{2k}u - |v|^{2k}v\right) d\tau.
\end{aligned}
$$

With the above and Lemma 4.1.1 we have that

$$
\begin{aligned}
&||\mathscr{T}u - \mathscr{T}v||_{C([0,T],M^{s,\alpha}_{p,q})} \\
&= \sup_{0\le t\le T} ||\mathscr{T}u - \mathscr{T}v||_{M^{s,\alpha}_{p,q}} \\
&\le C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \left(||u_0 - v_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + \sup_{0\le t\le T} ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \right) \\
&= C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \sup_{0\le t\le T} \left(||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \right) \\
&\le C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \sup_{0\le t\le T} ||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} (2k+1)\mathcal{L}^{2k} \\
&\le \frac{1}{2} \sup_{0\le t\le T} ||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\
&= \frac{1}{2} ||u-v||_{C([0,T],M^{s,\alpha}_{p,q})}.
\end{aligned}
$$

This show that $\mathscr{T}$ is a contraction map on $B_{\mathcal{L}}$. Thus, by the fixed point theorem we have a unique solution in $B_{\mathcal{L}}$. □

4.2 Solution to the Nonlinear Generalized Half Klein Gordon Equation

Using the same ideas as in Section 4.1 we have a similar theorem for the generalized half Klein-Gordon equation.

Theorem 4.2.1. *(Trulen) Let $1 \le p,q \le \infty$, $s > s_0$, and $T \ge 1$. Suppose k is a* *positive integer and there is positive constant c_k dependent only on k such that*

$$||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{c_k}{T^{n|\frac{1}{p}-\frac{1}{2}|(1+\frac{1}{2k})}T^{\frac{1}{2k}}}.$$

Suppose $1 \leq \alpha_o \leq 2(1-\alpha)$, then the nonlinear generalized half Klein-Gordon equation

$$\begin{cases} i\partial_t u - (I-\Delta)^{\frac{\alpha_o}{2}}u + F(u) = 0, & \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \textit{for } x \in \mathbb{R}^n, \end{cases}$$

when $F(u) = |u|^{2k}u$ has a unique solution $u \in C([0,T], M^{s,\alpha}_{p,q})$.

Proof. The the nonlinear generalized half Klein-Gordon equation has an equivalent form

$$u(t,\cdot) = e^{it(I-|\Delta|)^{\frac{\alpha_o}{2}}}u_0 - \int_0^t e^{i(t-\tau)(I-\Delta)^{\frac{\alpha_o}{2}}}F(u(\tau,\cdot))d\tau.$$

Consider the mapping

$$\mathscr{T}_K u = e^{it(I-\Delta)^{\frac{\alpha_o}{2}}}u_0 - \int_0^t e^{i(t-\tau)(I-\Delta)^{\frac{\alpha_o}{2}}}F(u(\tau,\cdot))d\tau.$$

Let C_j where $j = 1,2,3$ denote some positive constants that are independent of all essential variables. By Theorem 3.1.3 and by the same argument as in Theorem 4.1.2 the result follows. □

4.3 Solution to the Nonlinear Wave Equation

Theorem 4.3.1. *(Trulen) Let $p = 2$, $1 \leq q \leq \infty$, $s > s_w$ where s_w is defined as*

$$s_w = \begin{cases} \dfrac{n\alpha}{2}, & \textit{when } p = 2 \textit{ and } q = 1, \\ \dfrac{n(2q+2\alpha-\alpha q-2)}{2q}, & \textit{when } p = 2, \end{cases}$$

$T \geq 2$, k be a positive integer, and there is positive constant c_k depended only on k such that

$$||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{c_k}{T^{\frac{1}{k}}},$$

and

$$||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{c_k}{T^{\frac{1}{k}+1}},$$

then nonlinear wave equation

$$\begin{cases} \partial_{tt}u - \Delta u + F(u) = 0, & \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f_u(x), & \textit{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = g_u(x), & \textit{for } x \in \mathbb{R}^n, \end{cases}$$

where $F(u) = |u|^{2k}u$ *has a unique solution* $u \in C([0,T], M^{s,\alpha}_{2,q}(\mathbb{R}^n)$.

Proof. Note that $M^{s,\alpha}_{2,q}(\mathbb{R}^n)$ is a multiplication algebra when $s > s_w$.

Then the wave equation has a formal solution of the form

$$u(t,x) = \cos(t(-\Delta)^{\frac{1}{2}})f_u(x) + \Theta(t)g_u(x) - \int_0^t \Theta(t-\tau)F(u(\tau,x))d\tau.$$

Consider the map

$$\mathscr{T}_w u(t,x) = \cos(t(-\Delta)^{\frac{1}{2}})f_u(x) + \Theta(t)g_u(x) - \int_0^t \Theta(t-\tau)F(u(\tau,x))d\tau.$$

Let C_j where $j = 1,2,3$ denote constants that are independent of all essential variables. Then it follows from Theorems 3.1.1 and 3.1.4 that

$$\begin{aligned} &\left|\left|\cos(t(-\Delta)^{\frac{1}{2}})f_u + \Theta(t)g_u\right|\right|_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \\ &\leq C_1 \left(||f_u||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} + (1+t)\, ||g_u||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \right). \end{aligned}$$

Also, by multiplication algebra property of α-modulation space $M^{s,\alpha}_{2,q}(\mathbb{R}^n)$ we have

$$\left|\left||u(t,\cdot)|^{2k+1}\right|\right|_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \leq A_{2k+1}\, ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)}.$$

Then it follows that

$$\begin{aligned} &\left|\left|\int_0^T \frac{\sin\left((t-\tau)(-\Delta)^{\frac{1}{2}}\right)}{-\Delta)^{\frac{1}{2}}} F(u(\tau,\cdot))d\tau\right|\right|_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \\ &\leq C_2 \int_0^T (1+(t-\tau)) \left|\left||u(\tau,\cdot)|^{2k}u(\tau,\cdot)\right|\right|_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} d\tau \\ &\leq C_2 M_k T^2\, ||u(\tau,\cdot)||^{2k+1}_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)}, \end{aligned}$$

where $M_k = \max\{A_{2k}, A_{2k+1}\}$. Thus it follows that

$$\begin{aligned}
&||\mathscr{T}_w u||_{C([0,T],M_{2,q}^{s,\alpha}(\mathbb{R}^n))} \\
&= \sup_{0\le t\le T} ||\mathscr{T}_w u||_{M_{2,q}^{s,\alpha}(\mathbb{R}^n)} \\
&\le C_3\left(||f_u||_{M_{2,q}^{s,\alpha}(\mathbb{R}^n)} + T\,||g_u||_{M_{2,q}^{s,\alpha}(\mathbb{R}^n)} + T^2 \sup_{0\le t\le T} ||u(t,\cdot)||_{M_{2,q}^{s,\alpha}(\mathbb{R}^n)}^{2k+1}\right).
\end{aligned}$$

Now define $\mathcal{L}_w$ by

$$\mathcal{L}_w = \frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{\frac{1}{2k}}T^{\frac{1}{k}}},$$

and let $B_{\mathcal{L}_w}$ be the closed ball of radius $\mathcal{L}_w$ centered at the origin in the space $C([0,T], M_{2,q}^{s,\alpha}(\mathbb{R}^n)$.

Now suppose that

$$||f_u||_{M_{2,q}^{s,\alpha}(\mathbb{R}^n)} \le \frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}}},$$

and

$$||g_u||_{M_{2,q}^{s,\alpha}(\mathbb{R}^n)} \le \frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}+1}}.$$

Now it follows that

$$
\begin{aligned}
&||\mathscr{T}_w u||_{C([0,T],M^{s,\alpha}_{2,q}(\mathbb{R}^n))} \\
&= \sup_{0\le t\le T} ||\mathscr{T}_w u||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \\
&\le C_3 \left(||f_u||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} + T\,||g_u||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} + T^2 \sup_{0\le t\le T} ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \right) \\
&\le C_3 \left(\frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}}} + \frac{T}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}+1}} \right. \\
&\qquad \left. + T^2 \left(\frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{\frac{1}{2k}}T^{\frac{1}{k}}} \right)^{2k+1} \right) \\
&= C_3 \left(\frac{2}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}}} + \frac{T^2}{(2k+1)^{1+\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{2+\frac{1}{k}}} \right) \\
&\le \frac{3C_3}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}}} \\
&= \frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{\frac{1}{2k}}T^{\frac{1}{k}}},
\end{aligned}
$$

therefore $\mathscr{T}_w : \mathcal{L}_w \to \mathcal{L}_w$. Furthermore, it follows that

$$
\begin{aligned}
\mathscr{T}_w u - \mathscr{T}_w v = \cos(t(-\Delta)^{\frac{1}{2}})(f_u - f_v) + \Theta(t)(g_u - g_v) \\
- \int_0^t \Theta(t-\tau)(F(u(\tau,\cdot)) - F(u(\tau,\cdot)))d\tau.
\end{aligned}
$$

Again by Lemma 4.1.1 and the fact that we have the hypothesis

$$
||g||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \le \frac{1}{(2k+1)^{\frac{1}{2k}}(3C_3)^{1+\frac{1}{2k}}T^{\frac{1}{k}+1}},
$$

which implies that

$$
||g||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \le ||f||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)},
$$

it follows that

$$
\begin{aligned}
&||\mathscr{T}_w u - \mathscr{T}_w v||_{M^{a,\alpha}_{2,q}(\mathbb{R}^n)} \\
&\leq C_3 \Big(||f_u - f_v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} + (1+t)\,||g_u - g_v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \\
&\qquad + T^2 \sup_{0\leq t\leq T} ||u-v||^{2k+1}_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \Big) \\
&\leq C_3 \left((2+t)\,||f_u - f_v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} + T^2 \sup_{0\leq t\leq T} ||u-v||^{2k+1}_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \right) \\
&\leq C_3 T^2 \sup_{0\leq t\leq T} \left(||u-v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} + ||u-v||^{2k+1}_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \right) \\
&\leq C_3 T^2 \sup_{0\leq t\leq T} ||u-v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)}\,(2k+1)\mathcal{L}_w^{2k} \\
&\leq C_3 T^2 \frac{2k+1}{(2k+1)(3C_3)T^2} \sup_{0\leq t\leq T} ||u-v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \\
&\leq \frac{1}{3} \sup_{0\leq t\leq T} ||u-v||_{M^{s,\alpha}_{2,q}(\mathbb{R}^n)} \\
&= \frac{1}{3}\, ||u-v||_{C([0,T],M^{s,\alpha}_{2,q}(\mathbb{R}^n))}.
\end{aligned}
$$

Thus $\mathscr{T}_w$ is a contraction mapp and by the fixed point theorem we have a unique solution in the space $C([0,T], M^{s,\alpha}_{2,q}(\mathbb{R}^n))$. □

4.4 Solution to the Nonlinear Klein Gordon Equation

Using the same idea as in Section 4.3 we have a similar theorem for the Klein Gordon equation.

Theorem 4.4.1. *(Trulen) Let* $1 \leq p, q \leq \infty$, $t \geq 1$, $\alpha \leq \min\left\{\frac{1}{2}, \frac{2}{n}\right\}$, $s > s_0$ *where* s_0 *has been defined early. Suppose* k *is a positive integer and there exists a constant* c_k *that is depended on* k *only such that*

$$||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{c_k}{T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)} T^{\frac{1}{2k}}},$$

and

$$||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{c_k}{T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)} T^{1+\frac{1}{2k}}},$$

then the Nonlinear Klein-Gordon equation

$$\begin{cases} \partial_{tt}u(t,x) + u(t,x) - \Delta u(t,x) + F(u(t,x)) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f_u(x), & \text{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = g_u(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $F(u(t,x)) = |u(t,x)|^{2k}u(t,x)$ *has a unique solution* $u \in C([0,T], M^{s,\alpha}_{p,q}(\mathbb{R}^n))$.

Proof. Let C_j where $n = 1,2,3$ are all essential constants that are independent of all essential variables. The nonlinear Klein-Gordon Equation has the following formal solution

$$\begin{aligned} u(t,x) &= \cos(t(I-\Delta)^{\frac{1}{2}})f_u(x) + \Theta_K(t)g_u(x) \\ &\quad - \int_0^t \Theta_K(t-\tau)F(u(\tau,x))d\tau. \end{aligned}$$

Thus define the map $\mathscr{T}_{KG}$ by

$$\begin{aligned} \mathscr{T}_{KG}u &= \cos(t(I-\Delta)^{\frac{1}{2}})f_u(x) + \Theta_K(t)g_u(x) \\ &\quad - \int_0^t \Theta_K(t-\tau)F(u(\tau,x))d\tau. \end{aligned}$$

By the previous theorems and hypothesis we have

$$\begin{aligned} &\left|\left|\cos(t(I-\Delta)^{\frac{1}{2}})f_u + \Theta_K(t)g_u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\ &\le C_1\left(t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\,||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}(1+t)\,||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\left|\left|\int_0^t \Theta_K(t-\tau)F(u(\tau,x))d\tau\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\ &\le C_2\int_0^T \left((1+(t-\tau))^{n\left|\frac{1}{p}-\frac{1}{2}\right|} + (1+(t-\tau))^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\right)\left|\left||u|^{2k}u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}d\tau \\ &\le C_2M_k\left(T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} + T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\right)\left|\left||u|^{2k}u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\ &\le C_2M_kT^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}(1+T)\sup_{0\le t\le T}||u||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\ &\le C_2M_k'T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\sup_{0\le t\le T}||u||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}, \end{aligned}$$

where M_k' is defined by

$$M_k' = 2\max\{A_{2k}, A_{2k+1}\}.$$

Thus we have

$$\begin{aligned}
&||\mathscr{T}_{KG}u||_{C([0,T],M_{p,q}^{s,\alpha}(\mathbb{R}^n))}\\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\left(||f_u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} + T\,||g_u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} + T^2 \sup_{0\leq t\leq T}||u(t,\cdot)||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)}^{2k+1}\right).
\end{aligned}$$

Now define $\mathcal{L}_{KG}$ by

$$\mathcal{L}_{KG} = \frac{1}{(3C_3)^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}\left(T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\right)^{\frac{1}{2k}}},$$

and define $B_{\mathcal{L}_{KG}}$ be an open ball centered at the origin in $C([0,T],M_{p,q}^{s,\alpha}(\mathbb{R}^n))$ with radius $\mathcal{L}_{KG}$. Suppose that the following estimates hold

$$||f_u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \leq \frac{1}{(3C_3)^{1+\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)}T^{\frac{1}{k}}},$$

and

$$||g_u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \leq \frac{1}{(3C_3)^{1+\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)}T^{\frac{1}{k}+1}}.$$

So it follows that

$$\begin{aligned}
&||\mathscr{T}_{KG}u||_{C([0,T],M_{p,q}^{s,\alpha}(\mathbb{R}^n))}\\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\left(||f_u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} + T\,||g_u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} + T^2 \sup_{0\leq t\leq T}||u(t,\cdot)||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)}^{2k+1}\right)\\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\frac{3}{(3C_3)^{1+\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)}T^{\frac{1}{k}}}\\
&= \frac{1}{(3C_3)^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}\left(T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\right)^{\frac{1}{2k}}}.
\end{aligned}$$

Therefore, $\mathscr{T}_{KG} : B_{\mathcal{L}_{KG}} \to B_{\mathcal{L}_{KG}}$. Furthermore, we have

$$\begin{aligned}
\mathscr{T}_{KG}u - \mathscr{T}_{KG}v = {}& \cos(t(I-\Delta)^{\frac{1}{2}})(f_u(x)-f_v(x)) + \Theta_K(t)(g_u(x)-g_v(x))\\
&- \int_0^t \Theta_K(t-\tau)(F(u(\tau,x)) - F(v(\tau,x)))d\tau.
\end{aligned}$$

Now using the hypothesis we have

$$||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq ||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)},$$

we have

$$\begin{aligned}
&||\mathscr{T}_{KG}u - \mathscr{T}_{KG}||_{C([0,T],M^{s,\alpha}_{p,q}(\mathbb{R}^n))} \\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\Big(||f_u - f_v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T\,||g_u - g_v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\
&\qquad + T^2 \sup_{0\leq t\leq T} ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\Big) \\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\left((2+t)\,||f_u - f_v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T^2 \sup_{0\leq t\leq T} ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right) \\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2} \sup_{0\leq t\leq T}\left(||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right) \\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2} \sup_{0\leq t\leq T} ||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} (2k+1)\mathcal{L}^{2k}_{KG} \\
&\leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2} \frac{2k+1}{3c_3(2k+1)T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}} \sup_{0\leq t\leq T} ||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\
&\leq \frac{1}{3} \sup_{0\leq t\leq T} ||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \\
&= \frac{1}{3}\,||u-v||_{C([0,T],M^{s,\alpha}_{p,q}(\mathbb{R}^n))},
\end{aligned}$$

therefore $\mathscr{T}_{KG}$ is a contraction map and by the fixed point theorem there exists a unique solution $u \in C([0,T], M^{s,\alpha}_{p,q}(\mathbb{R}^n))$. □

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